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Asymptotic Structure and Symmetries of FLRW Universes

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"You've got to learn to do monkey tricks!"

Abstract

The asymptotic structure and symmetries of asymptotically flat spacetime are closely related to the gravitational memory effect, whereby detectors are permanently displaced relative to one another due to the passing of gravitational radiation. We examine the asymptotic properties and structure of a class of non-asymptotically flat spacetimes, including the dust-filled spatially hyperbolic and decelerating spatially flat Friedmann–Lemaître–Robertson–Walker universes. In order to study asymptotic structure, we inspect the Bondi–Sachs criterion of asymptotic flatness with respect to these universes, and compare the rate of falloff in the deviation of outgoing radial light rays. We take the first steps toward studying gravitational memory in these universes by deriving their associated groups of asymptotic Killing vectors, and comparing them to the well-known case of asymptotically flat spacetime.

Contents

1 Introduction

An ancient and foundational question in physics is of the nature of space and time. Before the advent of special and general relativity at the turn the twentieth century, spacetime was understood to be the ambient, non-dynamical background on which the physical systems of nature play out. The modern understanding provided by general relativity is that spacetime itself is a dynamical manifold coupled to the matter and energy it contains. This dual role of spacetime introduces a layer of complexity which prevents widely accepted definitions of energy in general relativity—in particular, of gravitational energy.

The presence of symmetries in a given spacetime enable the formulation of concrete energy–momentum conservation laws, and these are therefore of central importance. However, such symmetry-dependent laws do not exist for general spacetimes. Analogously, gravitational waves and the gravitational memory effect can be elegantly understood in terms of spacetime symmetries manifested asymptotically far from gravitational sources. However, the usual analysis [1–5] is largely concerned with the class of spacetimes which are stationary and flat at large distances from sources, and does not apply to cosmologies with a finite past nor non-zero spatial curvature, for example. This project aims to investigate the asymptotic structure of simple spacetimes which need not be flat at asymptotically large distances from sources, with the intention of extending the usual analysis of asymptotic symmetries and gravitational memory to more general spacetimes.

Gravitational Waves

Gravitational waves are perturbations in spacetime propagating at the speed of light which are caused by accelerated masses. Gravitational waves carry energy, causing binary systems to in-spiral as orbital energy escapes in the form of gravitational radiation. The first direct detection of gravitational waves was made by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2015 [6], where gravitational waves emitted by the in-spiral and final merger of a binary black hole system were measured on Earth for the first time.

Historically, there was debate over the physical reality of gravitational waves as solutions to Einstein's field equations. Gravitational waves were predicted in 1916 by Einstein in the context of linearised gravity, in which the perturbations of spacetime are infinitesimal and can be modelled by a field on a non-dynamical background spacetime. For several decades, it was unclear whether the wave-like solutions in linearised gravity were physical, or merely artefacts of linearisation [7]. Furthermore, the idea of energy-carrying gravitational waves was problematic, since gravitational energy is not localisable. This is because of Einstein's strong equivalence principle, which asserts that there always exists a local inertial frame in which the presence of gravity is indistinguishable from its absence; i.e., in which physical laws reduce to special relativity within a sufficiently small neighbourhood. This means a frame-invariant definition of local gravitational energy must imply that gravitational energy vanishes everywhere. Hence, a sensible definition of gravitational energy is non-local.

Motivated by the need to distinguish between physical, observer-independent gravitational wave phenomena and 'coordinate illusions', H. Bondi, K. Sachs and others developed a framework to describe gravitational radiation in an invariant fashion [1, 8], nearly half a century after Einstein's initial proposal of gravitational waves. This framework is the Bondi–Sachs formalism. The framework is well suited to the study of gravitational waves, since it coordinatises spacetime using the null (lightlike) geodesics followed by outgoing gravitational radiation. It also provided the first convincing evidence that gravitational radiation was physical and accompanied by energy loss in isolated systems. The formalism also provides a natural way to characterise the asymptotic flatness of a spacetime, and furthermore, its asymptotic symmetries.

Asymptotic Symmetries

The symmetries which arise at large distances in asymptotically flat spacetimes are important for understanding gravitational waves. While an arbitrary spacetime possesses no bulk symmetries, many physically relevant spacetimes admit asymptotic symmetries which pertain to global structure. These asymptotic symmetries were first investigated by Bondi, Sachs et al. for the case of asymptotically flat spacetimes, characterised by the condition that the deviations of the physical metric from the flat metric fall off at an appropriate rate as on recedes from gravitational sources (detailed in section 3.1). Previously, it was assumed that the group of symmetries which left the asymptotic region invariant (i.e., which preserved the falloff condition for asymptotic flatness) was the same symmetry group as that of flat spacetime. However, Bondi et al. showed that this assumption was wrong: in fact, the asymptotic symmetry group—now known as the Bondi–Metzner–Sachs (BMS) group, detailed in chapter 5—is an infinite dimensional generalisation of the group of symmetries of flat spacetime [1, 3, 8]. It has since been established [9] that the enlargement of the asymptotic symmetry group can be attributed directly to the existence of gravitational waves.

This surprising discovery has a number of physical consequences. Firstly, the existence of an enlarged asymptotic symmetry group means that general relativity does not, in fact, reduce to special relativity far away from sources as was previously expected. Unlike special relativity, general relativity admits degenerate vacua (physically distinct vacuum states) which can even carry non-zero angular momentum [3, § 1.2]. Secondly, the enlarged symmetry group implies the existence of conserved charges left by gravitational radiation, which directly relate to the relatively new (c. 1974, [10]) phenomenon of gravitational memory.

Gravitational Memory

Gravitational memory is the physical effect whereby free-falling particles are permanently displaced relative to one another after the passage of a gravitational wave train—without

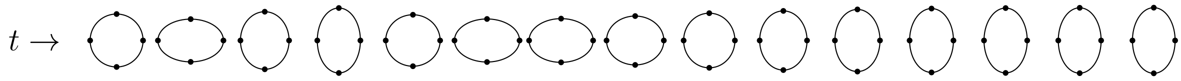


Figure 1.1: Exaggerated depiction of the gravitational memory effect in a simple detector. As a gravitational wave passes into the plane of the page, the proper distances among unaccelerated test particles fluctuate before remaining altered.

the particles being accelerated. (See figure 1.1. For a first introduction, see [11, 12].) The gravitational memory effect is of special empirical and theoretical interest, with the prospect of gravitational memory becoming detectable in the coming decades. It is generally agreed that LIGO is unable to detect gravitational memory (because of the incredible precision required and the problem of seismic noise), but next-generation instruments such as the space-bound Laser Interferometer Space Antenna (LISA) or the use of pulsar timing arrays show promise [12, § 1].

The memory effect arises as a consequence of gravitational radiation inducing transitions among the many degenerate vacua of spacetime. In initially flat spacetime, relatively stationary particles will oscillate[1] during the passing of a gravitational wave train before returning to a stationary configuration in the final vacuum state. However, since the initial and final vacua may be distinct, the particles' relative spacetime displacements may be permanently altered [4]. More formally, the initial and final spacetime configurations are related by the action of an asymptotic symmetry belonging to the BMS group (dubbed "supertransformations"[2]). This fascinating relationship between asymptotic symmetries and gravitational memory forms two vertices of what is known as "the infrared triangle of equivalence" [3, 13]. It is in this way that understanding the asymptotic symmetries of the BMS group lends valuable insight into the physical nature of gravitational memory.

Non-asymptotically Flat Spacetimes

A natural question which arises is how the asymptotic symmetries (and by extension, the memory effect) differ in spacetimes which are not asymptotically flat. Toward this end, we extend the analysis of the asymptotic structure and symmetries of flat spacetimes to various Friedmann–Lemaître–Robertson–Walker (FLRW) universes, which exhibit a finite past and zero or negative spatial curvature.

Chapter 2 explores the relationship between a simple non-gravitational model of an expanding universe (the Milne universe) and a particular FLRW model, which serves to provide insight into the asymptotic structure of the latter. In chapter 3, we investigate the asymptotic nature of various FLRW universes by inspecting their Bondi–Sachs flatness and the asymptotic deviation of light rays. Chapter 4 formally defines the asymptotic region of a spacetime via the Bondi–Penrose formalism, and makes the asymptotic structure of various FLRW universes explicit. Finally, asymptotic symmetries are defined and derived for selected FRLW spacetimes, and the results are compared to the asymptotically flat case. Sections 3.2, 3.3, 4.2, and 5.2 are original work.

[1]More precisely, it is the proper lengths between the particles (i.e., the metric itself) which fluctuates.

[2]"Supertransformations" bear no relation to supersymmetry; the prefix "super" is given to infinite-dimensional objects possessing familiar finite-dimensional analogues (in this case, the Poincaré group).

2 The Milne Universe

We are ultimately interested in the asymptotic properties of FLRW models as compared with Minkowski (flat) spacetime. The Milne universe is an interesting 'toy model' which bridges the gap between Minkowski spacetime and the future limit of the dust solution of the FLRW universe with negative spatial curvature. The Milne universe is equivalent to both under a suitable 'change of observers', providing a useful link between the two.

The Milne universe is a non-gravitational empty universe model proposed in 1935 by E. Milne [14] in an attempt to develop a rival theory to general relativity. Using this model, Milne purported to explain the expanding universe in the context of special relativity alone. While the Milne cosmology is not a successful modern alternative to Einstein's general relativity, its significance is that it is a non-gravitational model of an infinite, empty, expanding, negatively curved universe. Such features are often associated with gravitational cosmologies, and thus the Milne universe serves as useful example to help differentiate between those properties of a cosmological model which require gravity and those which are simply consequences of special relativity [15]. In this case, it is of interest because of its simplicity and resemblance in the infinite future to a particular FLRW universe in the gravitational theory, detailed in the following section.

The Milne universe can be constructed from empty Minkowski spacetime by introducing a 'big bang' at the origin from which an explosion of test particles are emitted over all possible directions and velocities $\beta \equiv |\boldsymbol{v}|/c < 1$. All matter is contained within the origin's future light cone $r < ct$, inside a finite spherical bubble expanding at the speed of light on a rigid Minkowski background. Spacetime at earlier times or beyond the region $r < ct$ where there exists no matter is interpreted as *non-physical* in the Milne model.

If the density of emitted test particles in velocity space is proportional to $\gamma^2 = (1-\beta^2)^{-1}$, then the Milne universe is Lorentz invariant about the origin. Thus, to an observer riding an inertial test particle (a *Milne observer*), the cosmological principle holds; i.e., the universe appears the same to any such Milne observer. This symmetry motivates the adoption of the Lorentz-invariant Milne observer proper time τ as the cosmic time coordinate. We also adopt the *reduced-circumference* radial coordinate $\mathfrak{r}$ to form Milne observer coordinates $(\tau, \mathfrak{r}, \Theta^A)$ where, in terms of concentric spherical coordinates (t, r, Θ^A),

$$c\tau = \sqrt{c^2t^2 - r^2}, \qquad \mathfrak{r} = r/c\tau, \tag{2.1}$$

are defined for $r < ct$ and the angular coordinates are preserved. While t-hypersurfaces of the Milne universe contain matter interior to the sphere of radius ct, τ-hypersurfaces contain matter spread homogeneously and isotropically to spatial infinity, resulting in an apparently open universe from the perspective of Milne observers. By substituting the

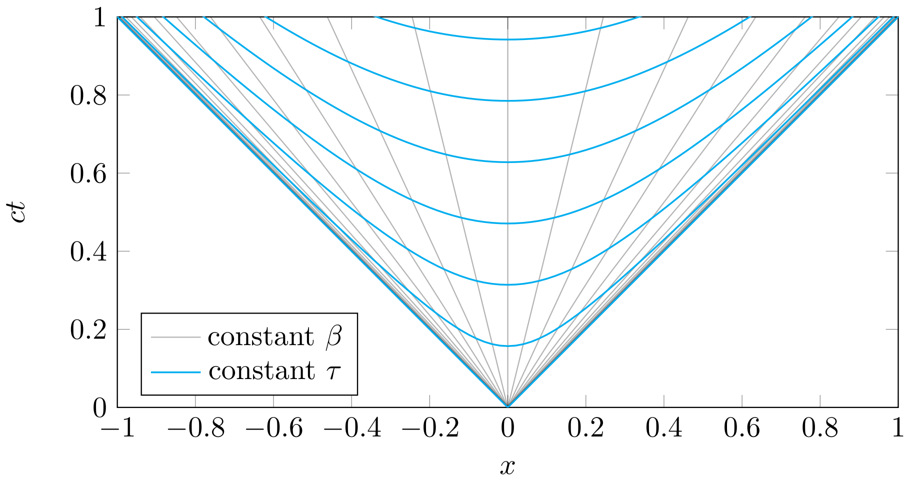


Figure 2.1: Depiction of the Milne universe embedded in flat Minkowski spacetime (t, x, y, z), with y and z suppressed. Milne observers are emitted from the origin and follow worldlines of constant β. Spacetime outside the cone $r < ct$ is not considered part of the Milne universe.

Milne coordinates (2.1) into the Minkowski metric with signature $(-, +, +, +)$, we derive the Milne metric $\boldsymbol{g} \equiv \mathrm{d}s^2$ in reduced-circumference spherical coordinates

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}\tau^2 + c^2\tau^2\left[\frac{\mathrm{d}\mathfrak{r}^2}{1+\mathfrak{r}^2} + \mathfrak{r}^2\mathrm{d}\Theta^2\right], \qquad \tau, \mathfrak{r} > 0, \tag{2.2}$$

where $\mathrm{d}\Theta^2 = \mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2$ is the metric of the unit 2-sphere, and $\mathrm{d}x^2 \equiv \mathrm{d}x\,\mathrm{d}x \equiv \mathrm{d}x \otimes \mathrm{d}x$. The Milne metric is so-called to emphasise the differing physical interpretation to the Minkowski metric, even though (2.2) is indeed isometric to (the future light cone of) Minkowski spacetime under the coordinate transformation (2.1).

The Milne metric may be written succinctly by employing *conformal time* η (satisfying $\mathrm{d}\tau = \tau\mathrm{d}\eta$) and *hyperspherical radial* χ coordinates,

$$\mathrm{d}s^2 = c^2\tau_0^2 e^{2\eta}\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sinh^2\chi\mathrm{d}\Theta^2\right], \tag{2.3}$$

where (η, χ) are dimensionless and related to the Milne coordinates by $\mathfrak{r} = \sinh\chi$ and $\tau = \tau_0 e^\eta$, with τ_0 as a temporal scaling constant. Throughout this report, the symbols η and χ will always refer respectively to the conformal time and hyperspherical radius associated with a metric.

2.1 The Milne Universe as an FLRW Universe

The Friedmann–Lemaître–Robertson–Walker (FLRW) metrics[1] are exact solutions to Einstein's field equations which describe a spatially homogeneous and isotropic expanding universe, first published in the 1920s [16–18]. In reduced-circumference spherical coordinates $(t, \mathfrak{r}, \Theta^A)$, a general FLRW metric has the form

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}\mathfrak{r}^2}{1-k\mathfrak{r}^2} + \mathfrak{r}^2\mathrm{d}\Theta^2\right], \tag{2.4}$$

[1]Also known as the Friedmann, FL, FRW, or RW metrics.

where $a(t)$ is the cosmic scale factor and the constant k is the Gaussian curvature of spacial sections when the scale factor is unity. The curvature constant—which may be positive, zero, or negative in the respective cases of closed, spatially flat or spatially hyperbolic universes—may be normalised to $k \in \{-1, 0, 1\}$ by rescaling $\mathfrak{r} \to \mathfrak{r}/\sqrt{|k|}$, which shall be assumed. Einstein's equations with the metric (2.4) imply the two[2] Friedmann equations, which together relate the scale factor to the pressure and energy density of the homogeneous universe. The first Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{c^2}{3}\Lambda, \tag{2.5}$$

where ρ is the matter density, Λ is the cosmological constant and $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$.

The Milne metric (2.2) has the form of the FLRW metric (2.4) with negative spatial curvature $k = -1$ and linear scale factor $a(\tau) = c\tau$. Linear cosmic expansion $a(t) \propto t$ is predicted by general relativity via the Friedmann equation (2.5) in the open case $k = -1$ precisely when the cosmological constant Λ and density ρ both vanish. Therefore, the Milne universe may be regarded as an empty, open FLRW universe with zero cosmological constant.

In a purely dust-filled universe, the density $\rho \propto a^{-3}$ scales as the inverse of spatial volume. In this case, the Friedmann equation (2.5) admits the parametric solution [17]

$$a = \mathscr{A}(\cosh\eta - 1), \qquad ct = \mathscr{A}(\sinh\eta - \eta), \tag{2.6}$$

where $\mathscr{A}$ is a constant of dimensions length and the parameter η is identified as conformal time, satisfying $c\,\mathrm{d}t = a\mathrm{d}\eta$. In terms of the conformal time coordinate η and hyperspherical radial coordinate $\chi = \operatorname{arcsinh}(\mathfrak{r}/\mathfrak{r}_0)$, the open FLRW metric (2.4) takes the form

$$\mathrm{d}s^2 = a^2\big[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Theta^2\big], \tag{2.7}$$

which differs from the Milne metric (2.3) only by the form of the scale factor. In the far future limit as $\eta \to \infty$, the scale factor of a dust-filled open FLRW universe $a \to \frac{1}{2}\mathscr{A}e^{\eta}$ approaches that of the Milne universe, $a(\tau) = c\tau_0 e^{\eta}$, if the constant of integration is chosen as $\mathscr{A} = 2c\tau_0$. In this sense, the Milne universe is obtained as the far future limit of a dust-filled open FLRW universe. This relationship allows for the consideration of properties of such an FLRW universe taken in the familiar Milne universe limit, where they may be more easily interpreted. It also suggests that there may be a sense in which the open FLRW universe is asymptotically flat to the future.

[2] The two Friedmann equations are related on account of the Bianchi identity; the first equation (2.5) and $\nabla_\mu G^{\mu\nu} = 0$ together imply the second (3.14).

3 Notions of Asymptotic Flatness

The flat Milne universe may be recovered from the open FLRW universe in the far future limit. We therefore expect that the open FLRW universe is 'eventually flat' in an appropriate sense. There exist several subtly distinct notions of asymptotic flatness.[1] Loosely speaking, this is because there is no a priori preferred method of determining what asymptotic boundary conditions should be applied to the dynamical manifold representing spacetime. Boundary conditions are typically chosen to be permissive enough to admit gravitational wave solutions, but strong enough to exclude non-physical spacetimes with infinite total energy [3, § 5.1]. In this section, aspects of the asymptotic flatness of the Milne, open FLRW and spatially flat FLRW universes are investigated from two approaches; the Bondi–Sachs criterion of asymptotic flatness and the behaviour of null geodesics at asymptotic distances.

3.1 The Bondi–Sachs Criterion

A simple and physically-motivated criterion of asymptotic flatness is due to Bondi and Sachs [1, 2]. The Bondi–Sachs criterion is a consequence of the requirement that the physical metric g_{ab} approach the Minkowski metric η_{ab} at a rate of $\mathcal{O}(1/r)$ for an appropriate radial coordinate r as one recedes along null directions from the origin. While it is a natural condition, it remains unclear whether this condition is too stringent to include other fields of interest [2]. The criterion itself employs retarded Bondi coordinates (u, r, Θ^A), which are defined to satisfy the following conditions [22, § 2]:

i) The coordinate u is a retarded null coordinate; a hypersurface of constant $u = u_0$ is the future light cone of the point $(r, u) = (0, u_0)$. Consequently, the normal covector $k_a = -\partial_a u$ to the u-hypersurface is null $(k^a k_a = 0)$ which implies $g^{uu} = 0$.

ii) Null rays of constant u are also of constant angular coordinate Θ^A. This implies the future-pointing tangent vector $k^a = g^{ab} k_b$ to a u-hypersurface satisfies $k^a \partial_a x^A = 0$, so that $g^{uA} = 0$.

iii) The radial coordinate r is an *areal coordinate*, i.e., $\det g_{AB} = r^4 \det q_{AB}$ where $q_{AB} \mathrm{d}x^A \mathrm{d}x^B = \mathrm{d}\Theta^2$ is the unit 2-sphere metric. Equivalently, $\partial_r \det (g_{AB}/r^2) = 0$.

[1]A spacetime may inequivalently be *asymptotically weakly simple* in the sense of Penrose [19], *asymptotically Minkowskian* in the sense of Geroch [20] or *asymptotically flat at null infinity* in the sense of Ashtekar–Xanthopoulos [21] to name a few. Note that *spatial* flatness does not imply *spacetime* flatness.

A general metric written in retarded Bondi coordinates, by virtue of the conditions above, has the form [1, pt. B]

$$\mathrm{d}s^2 = -\frac{V}{r}e^{2\beta}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + r^2 q_{AB}\big(\mathrm{d}x^A - U^A\mathrm{d}u\big)\big(\mathrm{d}x^B - U^B\mathrm{d}u\big), \tag{3.1}$$

where V, β and U^A are functions on spacetime and q_{AB} is the metric of the unit 2-sphere. Such a metric satisfies the *Bondi gauge* $g_{rr} = g_{rA} = 0$, allowing the Bondi form to be viewed as a (partial) gauge fixing of the metric components. A spherically symmetric spacetime, for which $U^A = 0$, then has the Bondi form

$$\mathrm{d}s^2 = -e^{2\beta}\left[\frac{V}{r}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r\right] + r^2\mathrm{d}\Theta^2. \tag{3.2}$$

Thus, for a spherically symmetric spacetime, the condition that r be an areal coordinate is equivalent to the requirement that the coefficient of the 2-sphere metric $\mathrm{d}\Theta^2$ be r^2.

The Bondi–Sachs criterion of asymptotic flatness itself is the conjunction of the boundary conditions

$$\lim \frac{V}{r} = 1, \qquad \lim \beta = \lim rU^A = 0, \qquad \lim h_{AB} = q_{AB},$$

where the limits are taken as $r \to \infty$ with u and Θ^A fixed [1]. It this limit, the general metric (3.1) reduces to Minkowski spacetime, and the criterion defines a concrete notion of asymptotic flatness—although they are a rather unsatisfactory replacement of more geometrical notions due to their explicit coordinate dependence [8, § 6].

3.1.1 The Bondi Form of a General FLRW Universe

We wish to express spherically symmetric FLRW metrics in Bondi form (3.2) so that we may inspect them in the Bondi–Sachs formalism. The general FLRW metric (2.4) in conformal time η and hyperspherical radial χ coordinates, for a normalised curvature k, is

$$\mathrm{d}s^2 = a^2(\eta)\big[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \mathrm{sink}(\chi)^2\mathrm{d}\Theta^2\big] \quad \text{where} \quad \mathrm{sink}(\chi) = \begin{cases} \sin\chi & k = +1, \\ \chi & k = 0, \\ \sinh\chi & k = -1. \end{cases}$$

In particular, this metric satisfies $\det g_{AB} = \left(a^2\,\mathrm{sink}(\chi)^2\right)^2 \det q_{AB}$, leading to the areal coordinate $r = a(\eta)\,\mathrm{sink}(\chi)$. Null coordinates are constant along null hypersurfaces $\mathrm{d}s^2 = 0 \iff \mathrm{d}\eta \pm \mathrm{d}\chi = 0$, and hence $u = \eta - \chi$ is a retarded null coordinate. These coordinates (u, r, Θ^A) yield the inverse Jacobian matrix

$$\begin{pmatrix}\mathrm{d}\eta \\ \mathrm{d}\chi\end{pmatrix} = \frac{1}{A+B}\begin{pmatrix} A & 1 \\ -B & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}u \\ \mathrm{d}r\end{pmatrix} \quad \text{where} \quad \begin{cases} A = a\,\mathrm{sink}'(\chi), \\ B = a\dot{a}\,\mathrm{sink}(\chi)/c, \end{cases} \tag{3.3}$$

leading to the Bondi form of a general FLRW universe:

$$\mathrm{d}s^2 = -\frac{a^2}{A+B}\big[(A-B)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r\big] + r^2\mathrm{d}\Theta^2. \tag{3.4}$$

By identification of (3.4) with the spherically symmetric metric (3.2), the Bondi–Sachs criterion for FLRW spacetimes reads

$$\lim\left(A - B\right) = \lim g_{ur} \equiv \lim \frac{-a^2}{A+B} = \mathcal{O}(1), \quad \text{with} \quad \begin{cases} r \to \infty \\ u = u_0 \end{cases}, \tag{3.5}$$

where $\mathcal{O}(1)$ denotes a non-zero constant with dimensions of length,[2] independent of r. Closed FLRW metrics with $k > 0$ will not be considered because of their finitude and lack of interesting asymptotic structure. From now on, an FLRW metric will refer only to spatially flat or open FLRW metrics.

3.2 Bondi–Sachs Flatness of FLRW Universes

We have shown that the isometrically flat Milne universe is the far future limit of the dust-filled open FLRW universe. With this relationship in mind, we are interested in whether such a universe satisfies the Bondi–Sachs criterion of asymptotic flatness to the far future: on the one hand, it is characterised by negative spatial curvature; on the other, it is asymptotic in the far future to the Milne universe.

We can easily verify that the Milne universe, with

$$a = c\tau_0 e^{\eta}, \qquad A = c\tau_0 e^{\eta}\cosh\chi, \qquad B = c\tau_0 e^{\eta}\sinh\chi,$$

satisfies the criterion (3.5), since $A - B = c\tau_0 e^{u_0} = \mathcal{O}(1)$ and $-\,g_{ur} = c\tau_0 e^{u_0} = \mathcal{O}(1)$. In the case of the dust-filled open FLRW universe, however, we find that

$$A = \mathscr{A}(\cosh\eta - 1)\cosh\chi, \quad \text{and} \quad B = \mathscr{A}\sinh\eta\sinh\chi, \tag{3.6}$$

leading to the limits

$$A - B = \mathscr{A}(\cosh u_0 - \cosh\chi) = \mathcal{O}(e^{\chi}) = \mathcal{O}(\ln r), \tag{3.7}$$

where $u_0 = \eta - \chi$, and

$$-g_{ur} = \frac{a^2}{A+B} = \frac{\mathscr{A}\left[\cosh\left(\frac{v+u_0}{2}\right) - 1\right]^2}{\cosh\left(\frac{v-u_0}{2}\right) - \cosh(v)} \to \frac{\mathscr{A}}{2}e^{u_0} = \mathcal{O}(1), \tag{3.8}$$

where $v = \eta + \chi \to \infty$. The second limit $-g_{ur} \to \mathscr{A}/2e^{u_0} = \mathcal{O}(1)$ is satisfied, while the first limit (3.7) diverges logarithmically (note that $r = a\sinh\chi = \mathcal{O}(e^{\eta+\chi}) \to \infty$). This indicates that the dust-filled open FLRW universe is not asymptotically flat in sense of Bondi–Sachs, even in the far future limit.

A related question is whether any spatially flat FLRW models satisfy the Bondi–Sachs criterion, or to what degree they fail. We consider spatially flat FLRW models with a single species of homogeneous, isotropic matter obeying a linear equation of state $p = w\rho c^2$,

[2] Note that the coordinates u and r in Bondi metric (3.1) as quoted from [1] have dimensions of length, whereas here u and r are dimensionless, with the metric components carrying dimensions of length instead.

where w is the dimensionless *parameter of state*. The case $w = 1/3$ is of particular interest, corresponding to a radiation-dominated cosmology (crudely) approximating our early universe. A spatially flat FLRW universe with such matter solves the Einstein equations if the scale factor satisfies the power law [17]

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} =: k_0 t^p. \tag{3.9}$$

The associated conformal time and areal coordinates are

$$c\,\mathrm{d}t = a \mathrm{d}\eta \iff \eta = \frac{1}{1-p}\frac{c}{k_0} t^{1-p}, \quad \text{and} \quad r = a\chi = k_0 \chi t^p. \tag{3.10}$$

The lengths A and B defined in (3.3),

$$A := \frac{\partial r}{\partial \chi} = k_0 t^p, \qquad\qquad B := \frac{\partial r}{\partial \eta} = \frac{k_0^2}{c} \chi p t^{2p-1},$$

lead to the limits

$$A - B = k_0 \left[(1-q)t^p + \frac{k_0}{c} p u_0 t^{2p-1}\right],$$

where $u_0 = \eta - \chi$ and $q := \frac{p}{1-p}$, and

$$g_{ur} = -\frac{a\eta}{(\eta - u_0)q + \eta}, \quad \text{where} \quad a = k_0 t^p = k_0 \left[\frac{k_0}{c}(1+q)\eta\right]^q. \tag{3.11}$$

The *strong energy condition*, derived from the requirement that 'matter must gravitate toward matter,' is the constraint that $w > -1/3 \iff 0 < p < 1$, corresponding in the spatially flat case to a decelerating universe. With the strong energy condition assumed, t grows monotonically with η, and the quantity $A - B$ may be evaluated in limit along outgoing null rays; $\eta, \chi \to \infty$, with $u = u_0$ fixed. The condition that $\lim(A-B) = \mathcal{O}(1)$ has two solutions: it is satisfied in the degenerate limit $p = 0 \iff a(t) = a_0$, corresponding to Minkowski spacetime; or in the case $q = 1 = 2p \iff w = 1/3$, corresponding to a radiation-filled universe. The second limit $\lim g_{ur} = \mathcal{O}(1)$ is satisfied in the degenerate case, but not in the radiation-dominated case, where $\lim g_{ur} = \mathcal{O}(\eta) = \mathcal{O}(\sqrt{r})$ diverges.

This shows that spatially flat FLRW universes are not asymptotically flat in the sense of Bondi–Sachs (except in the degenerate case of Minkowski spacetime); the criterion requires spatial *and temporal* asymptotic flatness. The extent to which these universes fail the criterion is apparent in the metric component $g_{ur} = \mathcal{O}(\eta^q) = \mathcal{O}\big(r^{q/2}\big)$. More interesting is the failure of the dust-filled open FLRW universe to satisfy the criterion (3.5), since its limit to the far future is exactly the Milne universe, whose metric is globally $\eta_{\mu\nu}$ in suitable coordinates. In this case, it is the metric component $g_{uu} = g_{ur}(A - B) = \mathcal{O}(\ln r)$ which diverges. Since $\mathcal{O}(\ln r)$ grows slower than $\mathcal{O}\big(r^{q/2}\big)$ for any $q > 0$, it appears that the open FLRW metric is 'closer' to satisfying the Bondi–Sachs criterion than any spatially flat FLRW metric—although this comparison is informal and its physical significance is unclear.

3.3 Asymptotic Deviation of Null Geodesics

Another indicator of asymptotic flatness is the behaviour of an initially near-parallel beam of light rays as they escape to infinity along null geodesics (i.e., to *future null infinity* $\mathscr{I}^+$, formally defined in chapter 4). More precisely, we are interested in the geodesic deviation of null rays at asymptotic distances. The geodesic deviation equation in the case of null geodesics implies Sachs' optical equation (3.12) [23], which is formulated in terms of a small area A orthogonal to the tangent vector $P^\mu := \mathrm{d}x^\mu/\mathrm{d}\lambda$ of the path $x^\mu(\lambda)$ of a null ray. The area A represents the cross section of a beam of co-moving light rays near $x^\mu(\lambda)$, and the growth of $A(\lambda)$ indicates divergence of the rays. Divergence is an indicator of spacetime curvature, and the rate at which divergence vanishes with respect to an affine parameter λ is a physical measure of 'how quickly a spacetime flattens out'.

With respect to an orthonormal frame $(P^\mu, Q^\mu, L_1^\mu, L_2^\mu)$ where Q^μ is a null vector orthogonal to P^μ and $L^\mu_{\alpha\in\{1,2\}}$ span the plane of A, Sachs' optical equation takes the form

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\sqrt{A} = -\frac{1}{2}\big(R_{\mu\nu}P^\mu P^\nu + w^2\big), \tag{3.12}$$

where $w^2 = w^{\alpha\beta}w_{\alpha\beta}$ is the square of the traceless shear of the null congruence [24, § 14]. In homogeneous and isotropic cosmologies, the Weyl curvature—and hence the shear tensor $w_{\alpha\beta}$—vanishes, so that the only contribution to divergence is from the Ricci tensor.

For a general FLRW metric (2.4) in reduced-circumference spherical coordinates (t, r, Θ^A), an outgoing radial null ray has tangent vector $P^\mu = c\,\mathrm{d}t/\mathrm{d}\lambda\,\big(1, \sqrt{1-kr^2}/ca, 0, 0\big)$ by the requirement $P_\mu P^\mu = 0$. The non-zero components of the Ricci tensor for a general FLRW spacetime are [25]

$$R_{tt} = -3\frac{\ddot{a}}{a}, \qquad R_{rr} = \frac{B}{1-kr^2}, \qquad R_{\theta\theta} = Br^2, \qquad R_{\phi\phi} = Br^2\sin^2\theta,$$

where $B = (a\ddot{a} + 2\dot{a}^2)/c^2 + 2k$. In this case, Sachs' optical equation (3.12) evaluates to

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{kc^2}{a^2}\right). \tag{3.13}$$

In the case of the Milne universe ($k = -1$, $a = ct$), the right hand side vanishes identically, since the Milne universe is everywhere flat. In the case of an FLRW universe, the first Friedmann equation (2.5), in conjunction with the second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \tag{3.14}$$

where p is the homogeneous pressure, reduce the optical equation (3.13) to

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} = -4\pi G\Big(\rho + \frac{p}{c^2}\Big)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2. \tag{3.15}$$

For a matter source obeying an equation of state $p = w\rho c^2$, the Friedmann equations relate the matter density to the scale factor by the power law

$$\rho = \rho_0\Big(\frac{a_0}{a}\Big)^{3(1+w)},$$

and the Einstein equations are satisfied if the scale factor satisfies the power law (3.9). In the spatially flat case, therefore, the optical equation (3.15) gives the rate of decay

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} = -4\pi G\rho_0(1+w)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left(\frac{t_0}{t}\right)^2 \propto \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{1}{t^2} \to 0,$$

for any $-1 < w < 1$, including dust $w = 0$ and radiation $w = 1/3$. We see that the asymptotic flatness of spatially flat FLRW universes is characterised by an $\mathcal{O}(1/t^2)$ falloff in the deviation of light rays.

On the other hand, the dust-filled open FLRW universe ($k = -1$, $p = 0$) is spatially curved with a matter density $\rho/\rho_0 = (a/a_0)^{-3}$ which decreases with increasing scale factor. The scale factor a and cosmic time ct as in (2.6) for such a universe both approach $\mathscr{A}/2e^\eta$ as $\eta \to \infty$, so that in the far future limit, $a/ct \approx 1$ and the optical equation (3.15) becomes

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} \propto \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{1}{t^3} \to 0.$$

We see that the $\mathcal{O}(1/t^3)$ falloff is faster in the open FLRW universe than in any spatially flat FLRW model: in a sense, it approaches flat space faster with increasing distance.

To better understand the physical meaning of the these falloff rates, we also consider the Schwarzschild solution as an example of an asymptotically Minkowskian spacetime. The geodesic deviation of a congruence of null geodesics with tangent vector P^μ is

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{d}\lambda^2} = R^\mu{}_{\rho\sigma\nu}P^\rho P^\sigma \xi^\nu,$$

where ξ^μ is the separation vector between nearby geodesics with affine parameter λ. Using spherical symmetry, the area A of a beam of light rays orthogonal to P^μ evolves as

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} \propto \frac{1}{\|\boldsymbol{\xi}\|}\frac{\mathrm{d}^2\|\boldsymbol{\xi}\|}{\mathrm{d}\lambda^2},$$

if the separation vector ξ^μ is taken to be in the plane of A with area $A \propto \|\boldsymbol{\xi}\| \equiv \xi^\mu\xi_\mu$. Furthermore, in Cartesian isotropic coordinates (t, x, y, z) with the ray travelling outward in the zt-plane, the separation vector $\boldsymbol{\xi} \in \operatorname{span}\{\partial_x, \partial_y\}$ is in the plane of A, and we may take it to be $\boldsymbol{\xi} = \xi^x\partial_x$ for simplicity. This results in

$$\frac{1}{\|\boldsymbol{\xi}\|}\frac{\mathrm{d}^2\|\boldsymbol{\xi}\|}{\mathrm{d}\lambda^2} = \frac{1}{\xi^x}\frac{\mathrm{D}^2\xi^x}{\mathrm{d}\lambda^2} = R^x{}_{\mu\nu x}P^\mu P^\nu \quad \text{(no sum on } x\text{)},$$

which, after computing the Riemann components (see appendix A), yields a falloff of $\mathcal{O}(1/r^3)$ for the Schwarzschild geometry. For null geodesics, this is equivalently $\mathcal{O}(1/t^3)$.

We have seen that the falloff of null geodesic deviation in a spatially flat FLRW universe is of a slower order, $\mathcal{O}(1/t^2)$, than in other spacetimes considered; namely, the open FLRW universe and the Schwarzschild geometry. The falloff in open FLRW universe, $\mathcal{O}(1/t^3)$, is the same as in the asymptotically flat Schwarzschild geometry, again suggesting that it is appropriate to extend notions of asymptotic flatness to include the open FLRW universe.

4 Asymptotic Regions of Spacetime

We wish to explore the asymptotic symmetries of the Milne and FLRW spacetimes. To do this, a concrete definition of the asymptotic region is beneficial. The asymptotic region may be identified as the boundary of a spacetime after it undergoes *conformal compactification*, i.e., an angle-preserving transformation onto a finite manifold possessing a boundary. This identification is a useful because such a transformation preserves the causal structure of the original spacetime; in particular, null vectors remain null. Penrose first published a conformal compactification of Minkowski spacetime in 1964 [19], resulting in the invention of Penrose (or Carter–Penrose) diagrams, allowing the study of infinite spacetimes in a finite context. The modern formulation of this technique, known as the Bondi–Penrose formalism [22], is outlined here.

If $\mathscr{M}$ is a smooth Lorentzian manifold equipped with a metric g_{ab}, then the pair $(\mathscr{M}, g_{ab})$ represents a spacetime of general relativity. In the Bondi–Penrose formalism, the physical metric $\hat{g}_{ab}$ on $\hat{\mathscr{M}}$ is conformally related to a non-physical metric g_{ab}, which exists on a compact spacetime manifold–with–boundary $\mathscr{M}$. In Penrose's compactification of Minkowski spacetime [19], null geodesics begin and terminate on subsets of the boundary $\partial\mathscr{M}$ of the compact spacetime known as *past* $\mathscr{I}^-$ and *future null infinity* $\mathscr{I}^+$, respectively. This enables the study of the symmetries of past or future null infinity $\mathscr{I}$, which enables an elegant formulation of the asymptotic behaviour of gravitational waves and memory.

Definition 1. A physical spacetime $(\hat{\mathscr{M}}, \hat{g}_{ab})$ is *asymptotic to* a compact spacetime $(\mathscr{M}, g_{ab})$, where $\mathscr{M}$ is a smooth manifold with boundary $\partial\mathscr{M} \equiv \mathscr{I}$, if and only if there exist:

i) a bijective mapping $\Psi : \hat{\mathscr{M}} \to \mathscr{M} \setminus \mathscr{I}$; and

ii) a smooth function $\Omega : \mathscr{M} \to \mathbb{R}$,

with the following properties:

a) $g_{ab} = \Omega^2 \hat{g}_{ab}$ everywhere on $\hat{\mathscr{M}}$;

b) $\Omega = 0$ and $\nabla_a \Omega \neq 0$ everywhere on $\mathscr{I}$.

The bijection Ψ is implicitly used in a) to bring the objects into the same space; explicitly, it reads $\Psi^* \boldsymbol{g} = \Psi^*(\Omega^2)\hat{\boldsymbol{g}}$ where Ψ^* is the pullback of Ψ. The condition $\nabla_a \Omega \neq 0$ on $\mathscr{I}$ ensures that the scalar field Ω may be used as a coordinate on $\mathscr{M}$, allowing Taylor expansions in Ω to be performed which capture the degree of fall-off of physical fields [7].

4.1 The Penrose Compactification of Flat Spacetime

The Penrose compactification of Minkowski spacetime [19] employs retarded $u = t - r$ and advanced $v = t + r$ null coordinates, (u, v, Θ^A). The Minkowski metric then reads[1]

$$\mathrm{d}\hat{s}^2 = -\mathrm{d}u\,\mathrm{d}v + \frac{1}{4}(u - v)^2 \mathrm{d}\Theta^2. \tag{4.1}$$

In the standard compactification, Minkowski spacetime $(\hat{\mathscr{M}}, \mathrm{d}\hat{s}^2)$ is mapped onto the manifold-with-boundary $\mathscr{M}$ by the diffeomorphism $\Psi : \hat{\mathscr{M}} \to \mathscr{M} \setminus \mathscr{I}$ defined by the coordinate mapping

$$(u, v, \Theta^A) \mapsto (U, V, \Theta^A) \quad \text{where} \quad \begin{cases} u = \tan U, \\ v = \tan V. \end{cases}$$

Because the radius r is non-negative, we have $u \leq v$, and the range of the compact coordinates may be written $(U, V) \in \mathbb{T}$, where $\mathbb{T}$ is a closed triangular subset of $\mathbb{R}^2$ given by $\mathbb{T} = \big\{(U, V) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \mid U \leq V\big\}$ (see figure 4.1). Thus, we may identify[2] the compact manifold with $\mathscr{M} \cong \mathbb{T} \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the 2-sphere. The induced metric on $\mathscr{M}$ (formally, the pushforward of $\hat{\boldsymbol{g}} \equiv \mathrm{d}\hat{s}^2$ by Ψ) is

$$\mathrm{d}s^2 = \Omega^2 \mathrm{d}\hat{s}^2 = -\mathrm{d}U\mathrm{d}V + \frac{1}{4}\sin^2(U - V)\mathrm{d}\Theta^2, \tag{4.2}$$

where the conformal factor

$$\Omega = \big[(1 + u^2)(1 + v^2)\big]^{-1/2} = \cos U \cos V,$$

is identified, as quoted in [19]. Note that Ω vanishes smoothly on $\mathscr{I}$, where u or v become infinite, so that the metric $\mathrm{d}s^2$ is well defined everywhere on $\mathscr{M}$. For convenience, define the compactified spherical coordinates $T = V + U$ and $R = V - U$ with ranges $-\pi \leq T \leq \pi$, $0 \leq R \leq |T|$. The boundary is then partitioned into spacelike, timelike and null infinity, according to figure 4.1.

future timelike	i^+	$(T, R) = (+\pi, 0)$
spatial	i^0	$(T, R) = (0, +\pi)$
past timelike	i^-	$(T, R) = (-\pi, 0)$
future null	$\mathscr{I}^+$	$(U, V) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \oplus \frac{\pi}{2}$
past null	$\mathscr{I}^-$	$(U, V) \in -\frac{\pi}{2} \oplus [-\frac{\pi}{2}, \frac{\pi}{2}]$

When considered in the context of the compactified spacetime $(\mathscr{M}, g_{ab})$, all globally timelike geodesics begin at past timelike infinity i^- and end at future timelike infinity i^+. Similarly, all globally spacelike geodesics terminate at spacelike infinity i^0. The fact that i^+, i^0 and i^- are single points in the diagram (representing 2-spheres in $\mathscr{M}$) corresponds to the vanishing of the coefficient of $\mathrm{d}\Theta^2$ in (4.2). (The manifold $\mathscr{M}$ does not possess

[1]Note that Penrose, employing the $(+, -, -, -)$ convention; writes (4.1) and (4.2) with opposite sign.

[2]Topologically, this is not an isomorphism unless one identifies each $\mathbb{S}^2$ along the coordinate singularity $r = 0$, or by using Cartesian spatial coordinates—but this is meant simply for visualisation.

singularities there; those points are coordinate singularities of the same type as the origin in polar coordinates.) Light rays, on the other hand, travel on null geodesics which originate at a point on $\mathscr{I}^-$ and terminate at a point on $\mathscr{I}^+$, travelling along diagonal lines [19]. Since gravitational radiation propagates along null rays, null infinity $\mathscr{I}$ and its symmetries are significant in the study of gravitational waves and memory.

4.2 The Compactification of FLRW Universes

We proceed to find a conformal compactification of an open FLRW universe, treating the Milne universe as a special case. As in the Penrose compactification of Minkowski spacetime, we employ null coordinates (u, v, Θ^A), defined by $u = \eta - \chi$ and $v = \eta + \chi$ where (η, χ) are the conformal time and hyperspherical radial coordinates, as in (2.7). The physical metric then takes the form

$$\mathrm{d}\hat{s}^2 = a^2(t)\left[-\mathrm{d}u\mathrm{d}v + \sinh^2\left(\frac{u-v}{2}\right)\mathrm{d}\Theta^2\right].$$

Note that the domain of the coordinate $t \in (0, \infty)$ affects the domain of the conformal time η, depending on the scale parameter $a(t)$ of the cosmology in question. If we define the compactification Ψ by

$$(u, v, \Theta^A) \mapsto (U, V, \Theta^A) \quad \text{where} \quad \begin{cases} \sinh u = \tan U, \\ \sinh v = \tan V. \end{cases}$$

then the induced metric on the compact manifold $\mathscr{M}$ is

$$\mathrm{d}s^2 = \Omega^2 \mathrm{d}\hat{s}^2 = -\mathrm{d}U\mathrm{d}V + \operatorname{sech} u \operatorname{sech} v \sinh^2\left(\frac{u-v}{2}\right)\mathrm{d}\Theta^2, \tag{4.3}$$

where $\Omega = a^{-1}(\cosh u \cosh v)^{-1/2}$ is the conformal factor. This particular choice of Ψ induces a metric $\mathrm{d}s^2$ which is finite everywhere on $\mathscr{M}$. The coefficient of $\mathrm{d}\Theta^2$ in (4.3) is bounded by one, and is zero at timelike infinity $i^\pm$. The compact manifold therefore possesses coordinate singularities at $i^\pm$ comparable to those in the Penrose compactification.

Considering the Milne universe (2.3) with $a(t) = ct$ and $t = t_0 e^\eta$, a physical singularity[3] is apparent at $t = 0$, corresponding to the Milne 'big bang'. The range of the coordinates is $(\eta, \chi) \in (-\infty, \infty) \times (0, \infty)$, and hence the range of the compact coordinates is $(U, V) \in \mathbb{T}$, as in the Penrose compactification. Again, the compact manifold has topology $\mathscr{M} \cong \mathbb{T} \times \mathbb{S}^2$ and the conformal diagram (figure 4.2a) resembles Minkowski spacetime. Unlike Minkowski spacetime, the region at $t = 0$ corresponding to $\mathscr{I}^-$ is singular, and hence the Milne universe possesses no physical past null infinity (from the perspective of Milne observers).

The dust-filled open FLRW universe with scale factor $a = \mathscr{A}(\cosh \eta - 1)$ has a similar physical singularity where $a = 0$ at $\eta = 0$. This restricts the compact coordinates to the triangular region $(U, V) \in \mathbb{T}' := \mathbb{T} \cap \{(U, V) \mid U + V \geq 0\}$, resulting in a compact manifold

[3] Regarding the Milne universe as a subset of Minkowski spacetime and suppressing its physical interpretation, this is merely a coordinate singularity.

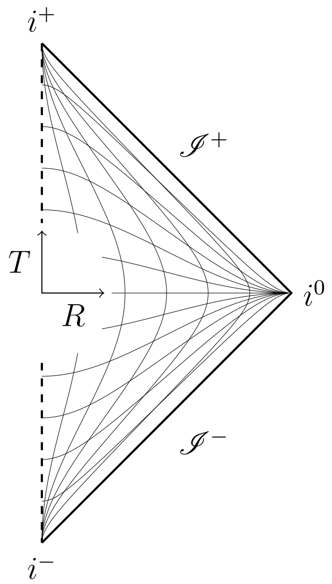
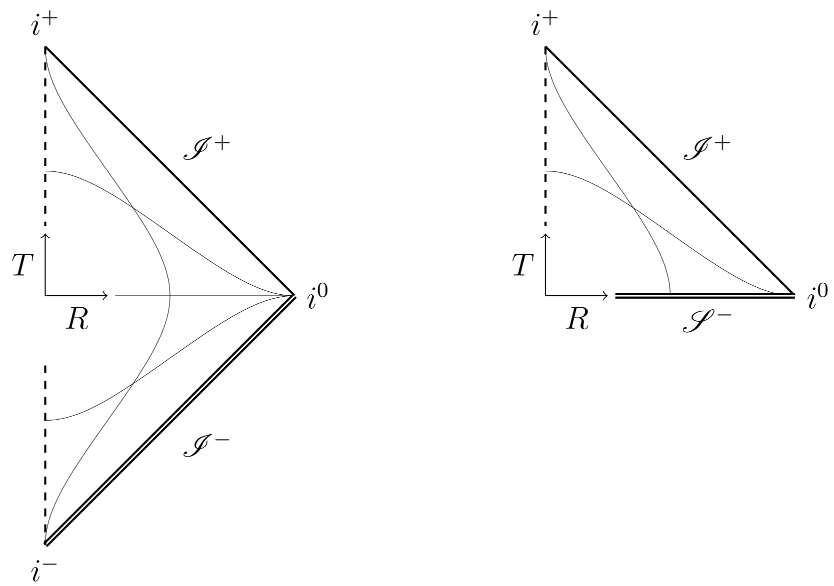


Figure 4.1: Penrose compactification of Minkowski spacetime. The two angular dimensions Θ^2 are suppressed, and the triangular region is $\mathbb{T}$. (Note that the dashed edge is not a boundary of $\mathscr{M}$.) The curved lines are timelike (vertical) and spacelike (horizontal) geodesics.



(a) Milne universe (b) ODF and DSF FLRW universes

Figure 4.2: Conformal diagrams of the Milne universe and a class of FLRW universes. Figure (b) is the conformal diagram of both the open dust-filled (ODF) FLRW universe and a decelerating, spatially flat (DSF) FLRW universe. Double-struck lines indicate singularities—i.e., regions which correspond to the same physical point in the original spacetime $(\hat{\mathscr{M}}, \hat{g}_{ab})$.

$\mathscr{M} \cong \mathbb{T}' \times \mathbb{S}^2$ with a spacelike singularity $\mathscr{S}^-$ to the past (figure 4.2b). This spacelike boundary $\mathscr{S}^-$ is *not* a past null infinity $\mathscr{I}^-$ because it intercepts timelike geodesics as well as null geodesics, and it is topology a point. Thus, the open dust-filled FLRW universes possesses only future null infinity $\mathscr{I}^+$, with topology $\mathbb{R} \times \mathbb{S}^2$ in $\hat{\mathscr{M}}$.

We shall also consider decelerating spatially flat FLRW spacetimes, which lends itself easily to conformal compactification when written in conformal time,

$$\mathrm{d}s^2 = a(\eta)^2\big[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \chi^2\mathrm{d}\Theta^2\big], \qquad a \propto \eta^q, \qquad q = \frac{2}{3w+1},$$

as the metric is manifestly conformal to Minkowski space with a conformal factor $a(\eta)$. The strong energy condition $w > -1/3$ is equivalent to the decelerating case $q > 0$ in a spatially flat universe. In the decelerating case, we simply quote from [5, § 2.2] the that the compactification may be performed with

$$(\eta, \chi, \Theta^A) \mapsto (T, R, \Theta^A) \quad \text{where} \quad (\eta, \chi) = \left(\frac{\sin T}{\cos T + \cos R}, \frac{\sin R}{\cos T + \cos R}\right),$$

with coordinate ranges $(T, R) \in \mathbb{T}'$. Consequently, the decelerating spatially flat FLRW universe also exhibits a future null infinity $\mathscr{I}^+$ and a spacelike singularity $\mathscr{S}^-$ to the past (figure 4.2b).

We have determined the asymptotic structures of the open FLRW and decelerating spatially flat FLRW universes, and shown that they both admit future null infinities $\mathscr{I}^+ \cong \mathbb{R} \times \mathbb{S}^2$ of the same topology as asymptotically flat space. We may now investigate in the asymptotic symmetries admitted by this class of spacetime at $\mathscr{I}^+$.

5 Asymptotic Symmetry Groups

In general relativity, a symmetry of spacetime (or simply, a *symmetry*) is an *isometry*; that is, a diffeomorphism[1] which preserves the metric. For instance, the symmetries of Minkowski spacetime form the Poincaré group, consisting of Lorentz transformations, spacetime translations and combinations thereof. Every continuous symmetry of a general spacetime can be represented as the flow of an associated vector field which *generates* the symmetry. (Mathematically, smooth vector fields form the Lie algebra to the Lie group of isometries.) Conversely, the diffeomorphisms generated as the flow of the vector field $\boldsymbol{\zeta} = \zeta^a \partial_a$ are symmetries if and only if the vector field satisfies Killing's equation,

$$\mathscr{L}_{\boldsymbol{\zeta}} g_{ab} = 0.$$

In the case of Minkowski spacetime, the flow of the constant vector field ∂_x is the continuous translation $(t, x, y, z) \mapsto (t, x + s, y, z)$ by the distance s. This translation is a symmetry because the metric is independent of x, so that $\mathscr{L}_{\partial_x} \eta_{ab} = 0$. By Noether's theorem, there exists an associated conserved charge: the x-component of relativistic four-momentum.

Asymptotic Killing vectors (AKVs) may be defined by a generalisation of Killing's equation, relaxing the condition of the vanishing of the metric under the Lie derivative to a set of boundary conditions that the metric and its perturbations are to satisfy. Explicitly, an asymptotic Killing vector $\boldsymbol{\zeta}$ satisfies

$$\mathscr{L}_{\boldsymbol{\zeta}} g_{ab} = \delta g_{ab} = \mathcal{O}(r^n), \tag{5.1}$$

where metric perturbations δg_{ab} are allowed to be of order $\leq n$ in the radial coordinate, where $n \in \mathbb{Z}$ is a falloff exponent, prescribed by the selected boundary conditions. In practice, it is convenient to introduce a (partial) gauge fixing such as the Bondi gauge, and require that the perturbations δg_{ab} also preserve the gauge. There is no canonical choice of boundary condition $\delta g_{ab} = \mathcal{O}(r^n)$ on the metric and its perturbations—even for the simplest case of asymptotically flat spacetime. For the reasons expounded in chapter 3, it is not clear which falloff conditions admit all physically interesting solutions while excluding all 'non-physical' ones. The process of choosing suitable falloff and gauge conditions to each spacetime has been humorously described as "more of an art than a science" [3, § 2.10].

[1] A *diffeomorphism* is a smooth transformation of spacetime, regarded as a mapping between manifolds of the same topology. The term is also used loosely by physicists to mean what we call an isometry.

5.1 Asymptotically Flat Spacetimes

In asymptotically Minkowski spacetime, the group of asymptotic Killing vectors of contains the Poincaré group as a subgroup, since Killing vectors are automatically AKVs. Before asymptotic symmetries were formally investigated in 1962 by Bondi et al. [1], it was assumed that the Poincaré group was indeed the entire asymptotic symmetry group of near-flat spacetime. However, to the surprise of Bondi, Metzner, van der Burg [1] and Sachs [2], the asymptotic symmetry group of near-flat spacetime (named the BMS group) is in fact an infinite-dimensional generalisation of the Poincaré group, as we will now see.

A general BMS 'supertransformation' can be found by solving the asymptotic Killing equation (5.1) subject to gauge–preservation and the boundary conditions prescribed by BMS. Preserving the Bondi gauge alone, $\delta g_{rr} = \delta g_{rA} = g^{AB}\delta g_{AB} = 0$, yields the general vector

$$\boldsymbol{\zeta} = f\partial_u + \left[-\frac{r}{2}\mathcal{D}_A Y^A + \frac{1}{2}\mathcal{D}^2 f + \mathcal{O}(1/r)\right]\partial_r + \left[Y^A - \frac{1}{r}\mathcal{D}^A f + \mathcal{O}\big(1/r^3\big)\right]\partial_A, \tag{5.2}$$

as derived in appendix B, where $f\big(u, x^A\big)$ and $Y^A\big(u, x^B\big)$ are arbitrary smooth functions and $\mathcal{D}$ is the covariant derivative on the 2-sphere. Constraining the form of (5.2) further by enforcing the boundary conditions [26]

$$g_{uu} = -1 + \mathcal{O}(1/r), \quad g_{ur} = -1 + \mathcal{O}\big(1/r^2\big), \quad g_{uA} = \mathcal{O}(1), \quad g_{AB} = r^2 q_{AB} + \mathcal{O}(r),$$

requires that $Y^A_{,u} = 0 \iff Y^A = Y^A\big(x^C\big)$ is a function of the 2-sphere, and that the u-dependence of f is of the form

$$f\big(u, x^C\big) = T\big(x^C\big) + \frac{u}{2}\mathcal{D}_A Y^A\big(x^C\big) + \mathcal{O}(1/r),$$

with $T\big(x^C\big)$ arbitrary.

5.1.1 The Subgroup of Supertranslations

The asymptotic Killing vector (5.2) of a perturbed Minkowski spacetime has three functions' worth of freedom; namely $T\big(x^C\big)$ and $Y^A\big(x^C\big)$. The Lie group of BMS supertransformations is therefore infinite-dimensional, possessing an infinite dimensional Lie algebra known as the *BMS algebra* [27, § 6.6]. The case where the Y^A vanish is of particular interest, wherein the AKV takes the simpler form

$$\boldsymbol{\zeta}_T = T\big(x^C\big)\partial_u + \frac{1}{2}\mathcal{D}^2 T\big(x^C\big)\partial_r - \frac{1}{r}\mathcal{D}^A T\big(x^C\big)\partial_A. \tag{5.3}$$

In the case that $T\big(x^C\big)$ is constant (an $\ell = 0$ spherical harmonic), equation (5.3) reduces to a constant translation in retarded time; $\boldsymbol{\zeta} \propto \partial_u$. Furthermore, the spatial translations $\partial_x, \partial_y, \partial_z$ are recovered when $T\big(x^C\big)$ is a linear combination of the three $\ell = 1$ spherical harmonics. When $T\big(x^C\big)$ consists of higher harmonics, $\ell \geq 2$, then $\boldsymbol{\zeta}$ is dubbed a *supertranslation.* Supertranslations can be viewed as angle-dependent translation in spacetime, and their action on vacua is in correspondence with gravitational memory induced by gravitational radiation [3, 9, 12, 26].

5.1.2 Asymptotic Symmetries and the BMS Group

Noether's theorem relates the conservation of the asymptotic structure of a spacetime to a set of associated *asymptotic symmetries*. It appears natural to define the group of asymptotic symmetries of a spacetime as exactly the group of diffeomorphisms generated by its AKVs, since such diffeomorphisms preserve asymptotic structure. However, distinct AKVs may act trivially on the asymptotic region; e.g., their action may be non-trivial on the interior spacetime while falling off as $r \to \infty$ at a rate sufficiently fast as to have no 'physical effect' at the boundary.[2] Thus, the group of AKVs possesses many AKVs for every physically distinct asymptotic symmetry.

Such gauge redundancy is undesirable in a definition of a symmetry group. Given a precise notion of sufficiently fast falloff (inherited from the choice of boundary conditions), AKVs can be classified as *proper* or *improper* according to whether their action on the boundary falls off sufficiently slowly or quickly, respectively. (Proper and improper AKVs are also respectively referred to as *trivial* and *large* asymptotic gauge transformations in, e.g., [26].) The gauge redundancy in the group of AKVs may be eliminated by taking the modulus over improper AKVs (a.k.a. trivial asymptotic gauge transformations). In the group theoretic language of [7], the asymptotic symmetry group is the quotient $\mathrm{Diff}_\infty(\mathscr{M})/\,\mathrm{Diff}^0_\infty(\mathscr{M})$ of the group $\mathrm{Diff}_\infty(\mathscr{M})$ of diffeomorphisms of the spacetime $\mathscr{M}$ which preserve the asymptotic boundary conditions by its subgroup $\mathrm{Diff}^0_\infty(\mathscr{M})$ of diffeomorphisms which are asymptotically trivial. The groups $\mathrm{Diff}_\infty(\mathscr{M})$ and $\mathrm{Diff}^0_\infty(\mathscr{M})$ consist of improper (large) and proper (trivial) AKVs, respectively.

With these technicalities aside, the group of AKVs of asymptotically flat spacetimes described in section 5.1 gives rise to the BMS group, $\mathscr{B}$. It has been shown that the BMS group has the natural semidirect[3] decomposition

$$\mathscr{B} \cong \mathrm{SO}(1,3) \ltimes \mathcal{S},$$

where $\mathrm{SO}(1,3)$ is the Lorentz group and $\mathcal{S}$ is group of *supertranslations*, which form an abelian subgroup of $\mathscr{B}$. In the case of the Poincaré group, $(\Lambda_1, \vec{T}_1) \bullet (\Lambda_2, \vec{T}_2) = (\Lambda_1\Lambda_2, \vec{T}_1 + \Lambda_1\vec{T}_2)$ where $\Lambda \in \mathrm{SO}(1,3)$ and $\vec{T} \in \mathbb{R}^{1+3}$. The BMS group may be understood as a direct generalisation of the Poincaré group, $\mathscr{P}$, which decomposes similarly into the Lorentz group and the abelian subgroup of spacetime translations, $\mathbb{R}^{1+3}$;

$$\mathscr{P} \cong \mathrm{SO}(1,3) \ltimes \mathbb{R}^{1+3}.$$

An alternative representation of the BMS group exploits the isomorphism of $\mathrm{SO}^+(1,3)$ with the group of Möbius transformations (conformal maps) $\mathrm{PGL}(2,\mathbb{C})$ on the Riemann sphere. With the celestial sphere coordinatised with the complex coordinate $z = e^{i\theta}\cot\frac{\theta}{2}$, a general BMS transformation is of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad u \mapsto K(z,\bar{z})u + \alpha(z,\bar{z}), \tag{5.4}$$

[2]The choice of boundary conditions determines which effects qualify as 'physical,' because whether or not an AKV vanishes 'sufficiently fast' is dependent on this choice [3, 26].

[3]The semidirect product, $\ltimes$, differs from the direct product by its modified group operation $\bullet : A \times B \to A \ltimes B$ defined by $(a_1, b_1) \bullet (a_2, b_2) = (a_1a_2, b_1\varphi_{a_1}b_2)$ where $\varphi : A \to \mathrm{Aut}(B)$ is an implicit homomorphism.

where $a, b, c, d \in \mathbb{C}$, $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \neq 0$ and K and α are functions on the celestial sphere [5, 28]. The isomorphism $\mathrm{SO}^+(1,3) \cong \mathrm{PGL}(2,\mathbb{C})$ may be interpreted as the statement that the orthochronous Lorentz group acts on the celestial 2-sphere in the same way that the Möbius group acts on the Riemann sphere; by conformal transformations [29]. Physically, this means that the effect of gravitational memory on a the vacuum is to leave it transformed by a combination of a Lorentz transformation and an angle-dependent time translation [3].

5.2 Asymptotically FLRW Cosmologies

Kehagias and Riotto [5] claim that decelerating spatially flat (DSF) FLRW cosmologies exhibit the full asymptotic BMS group on the basis that it possesses a future null infinity $\mathscr{I}^+$. Their argument is that one may always define a 'non-physical' metric g_{ab} in terms the physical metric $\hat{g}_{ab}$ by $g_{ab} = \hat{g}_{ab}/r^2$, before taking the degenerate $\lim_{r\to\infty} \mathrm{d}s^2 = \mathrm{d}\Theta^2$. For the physical metric $\mathrm{d}\hat{s}^2$ of a DSF FLRW universe, the non-physical metric is

$$\mathrm{d}s^2 = \frac{1}{a^2 r^2}\mathrm{d}\hat{s}^2 = \frac{1}{r^2}\left[-\mathrm{d}\eta^2 + \mathrm{d}r^2\right] + \mathrm{d}\Theta^2 \xrightarrow{r\to\infty} \mathrm{d}\Theta^2.$$

The 2-sphere metric $\mathrm{d}\Theta^2$ is then identified as the induced metric on $\mathscr{I}^+$, and its symmetries are identified as (5.4). Since the induced metric is independent of u, it also admits a general map $u \mapsto f(u, z, \bar{z})$ as a symmetry. However, Kehagias and Riotto impose a further restriction on this freedom by requiring the *null angle* $\mathrm{d}u/\mathrm{d}\Omega$ to be left invariant, where Ω is any solid angle. If, under conformal transformations of the 2-sphere, we have $\mathrm{d}\Theta \mapsto K(z,\bar{z})\mathrm{d}\Theta$, then to preserve the null angle $\mathrm{d}u/\mathrm{d}\Theta$, we demand similarly that $\mathrm{d}u \mapsto K(z,\bar{z})\mathrm{d}u$. This gives the general transformation $u \mapsto K(z,\bar{z})u + \alpha(z,\bar{z})$, where α is a constant of integration, as in (5.4) [5, § 3]. While this argument may be correct, it is much less explicit than the characterisation of asymptotic symmetries in terms of the asymptotic Killing equation.

We may extend the analysis of section 5.1 to calculate the asymptotic Killing vectors of perturbed FLRW metrics by enforcing equivalent boundary conditions. In the asymptotically flat case, the constraints on the metric perturbations as prescribed by BMS are

$$\delta g_{uu} = \mathcal{O}(1/r), \quad \delta g_{ur} = \mathcal{O}\big(1/r^2\big), \quad \delta g_{uA} = \mathcal{O}(1), \quad \delta g_{AB} = \mathcal{O}(r), \quad \delta g_{rr} = \delta g_{rA} = 0.$$

An equivalent set of boundary conditions for an asymptotically–FLRW universe must admit the same perturbative freedoms. Therefore, the Bondi form (3.4) of an FLRW universe, subject to equivalent boundary conditions has the metric components

$$\begin{aligned} g_{uu} &= -a^2\frac{A-B}{A+B} + \mathcal{O}(1/r), & g_{uA} &= \mathcal{O}(1), & g_{rr} &= 0, \\ g_{ur} &= -\frac{a^2}{A+B} + \mathcal{O}\big(1/r^2\big), & g_{AB} &= r^2 q_{AB} + \mathcal{O}(r), & g_{rA} &= 0. \end{aligned} \tag{5.5}$$

where $a = a(\eta)$ is the scale factor and $A(u,r)$ and $B(u,r)$ are as defined in section 3.1.1.

The metric (5.5) satisfies the Bondi gauge, so the analysis may begin in the same way as for an asymptotically flat metric. From the rr-component of equation (5.1), we obtain $\mathscr{L}_{\boldsymbol{\zeta}} g_{rr} = 2g_{ur}\zeta^u{}_{,r} = 0 \implies \zeta^u = f(u, x^A)$. The rA-components of (5.1) yield $\mathscr{L}_{\boldsymbol{\zeta}} g_{rA} = g_{AB}\zeta^B{}_{,r} + g_{ur}\zeta^u{}_{,A} = 0$, so that

$$\zeta^A{}_{,r} = -\frac{g_{ur}}{r^2}\mathscr{D}^A f + \mathcal{O}\big(1/r^4\big) \qquad \implies \qquad \zeta^A = Y^A\big(x^C\big) + \mathscr{D}^A f \int_r^\infty \mathrm{d}r' \, \frac{g_{ur'}}{(r')^2} + \mathcal{O}\big(1/r^3\big),$$

where $\mathscr{D}^A f = q^{AB} f_{,B}$. By virtue of the Bondi gauge, the r-component ζ^r arises identically to the asymptotically flat case, as derived in appendix B, $\zeta^r = -\frac{r}{2}\mathscr{D}_A Y^A + \frac{1}{2}\mathscr{D}^2 f + \mathcal{O}(1/r^3)$. These results yield the (the generator of) the equivalent BMS 'supertranslation' in an asymptotically FLRW universe,

$$\boldsymbol{\zeta}_T = f\partial_u + \frac{1}{2}\mathscr{D}^2 f \partial_r + \mathscr{D}^A f \int_r^\infty \mathrm{d}r' \, \frac{g_{ur'}}{(r')^2}\partial_A + \mathcal{O}\big(1/r^3\big), \tag{5.6}$$

where we have set $Y^A = 0$ and dropped $\leq \mathcal{O}(1/r^3)$ terms. The behaviour of the coefficient of $\mathscr{D}^A f$ as $r \to \infty$ is of particular interest: if it is of subleading order, then the action of the supertranslation $\boldsymbol{\zeta}_T$ on the celestial sphere vanishes; whereas if the term diverges, it may be an indication that the selected boundary conditions (5.5) are not well-suited to the spacetime in question.

5.2.1 Spatially Flat FLRW Cosmologies

With reference to (3.11), the metric of a perturbed spatially flat FLRW universe governed by the equation of state $p = w\rho c^2$ has ur-component

$$g_{ur} = -\frac{a\eta}{(\eta - u_0)q + \eta} + \mathcal{O}\big(1/r^2\big).$$

For simplicity, we restrict our attention to the celestial sphere at $u_0 = 0 \iff \eta = \chi$, in which case we find from (3.10) that $\eta = \chi = c^p r^{1-p}(1-p)^{-p}/k_0$, yielding an expression of g_{ur} in terms of r,

$$g_{ur} = -k_0(1-p)^{1+p}\Big(\frac{r}{c}\Big)^p.$$

Further abbreviating $h \coloneqq 1 - p = \frac{3w+1}{3(w+1)}$ and integrating with respect to r, we find that the angular components of the asymptotic Killing vector are

$$\zeta^A = Y^A + k_0\Big(\frac{c}{h}\Big)^{h-1}\frac{1}{r^h}\mathscr{D}^A f + \mathcal{O}\big(1/r^3\big).$$

Dropping $\mathcal{O}(1/r^3)$ terms, this leads to the spatially flat FLRW 'supertranslation'

$$\boldsymbol{\zeta}_T = f\partial_u + \frac{1}{2}\mathscr{D}^2 f\partial_r + C\frac{1}{r^h}\mathscr{D}^A f\partial_A, \tag{5.7}$$

where the coefficient C of dimensions (length)^h is $C = k_0\big(\frac{c}{h}\big)^{h-1} = a_0\big(\frac{1-p}{ct_0}\big)^p$.

The implication of this result is that, since $-1/3 < w \iff 0 < h < 1$, the $\mathscr{D}^A f$ terms in the ζ^A components fall off at a rate slower than $\mathcal{O}(1/r)$. Vector fields which are $\mathcal{O}(1)$ in r have angular components which are $\mathcal{O}(1/r)$, since $\partial_A = \mathcal{O}(r)$ for an areal coordinate r. Thus, we see that the norm of the supertranslation AKV grows as $\|\boldsymbol{\zeta}_T\| = \mathcal{O}\big(r^{1-h}\big) > \mathcal{O}(1)$, becoming infinite at $\mathscr{I}^+$, which suggests a breakdown of the usual analysis.

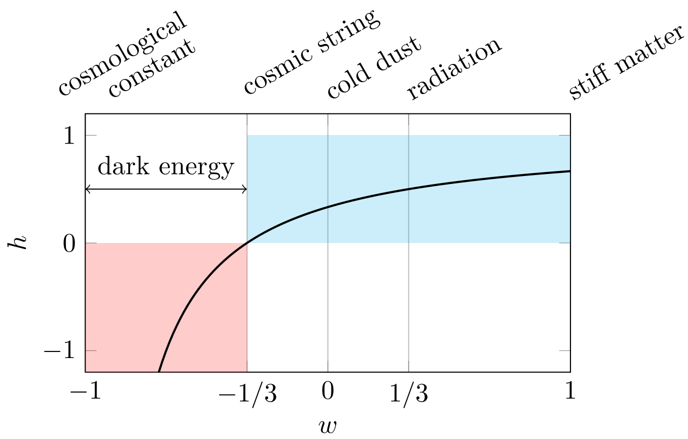


Figure 5.1: Relationship between equation of state parameter w and the exponent h of the $1/r$ falloff in the AKV (5.7) of a spatially flat FLRW spacetime. The AKV components ζ^A fall off slower than $1/r$ the $0 < h < 1$ region (blue) for all $w > -1/3$, and diverge for $-1 < w < -1/3$ (red).

5.2.2 The Open FLRW Cosmology

In the case of the open dust-filled FLRW universe, specified by (2.6) and (3.6),

$$a = \mathscr{A}(\cosh\eta - 1), \qquad ct = \mathscr{A}(\sinh\eta - \eta),$$
$$A = a\cosh\chi, \qquad B = \mathscr{A}\sinh\eta\sinh\chi,$$

the metric component g_{ur} is given by (3.8);

$$g_{ur} = -\frac{a^2}{A+B} = -\frac{\mathscr{A}(\cosh\eta - 1)^2}{\cosh(\eta+\chi) - \cosh\chi}.$$

We are interested in the integrand

$$\frac{g_{ur}}{r^2}\mathrm{d}r = -\frac{1}{\mathscr{A}}\operatorname{csch}^2\chi\,[\cosh(\eta+\chi) - \cosh\chi]^{-1}\,\mathrm{d}r,$$

which is simplified by the adoption of advanced $v = \eta + \chi$ and retarded $u = \eta - \chi$ null coordinates (v, u, x^A). Note that the AKV is evaluated at $\mathscr{I}^+$, attained when $r \to \infty$ with $u = u_0$ fixed. That is, the integration in (5.6) is along an outgoing null geodesic $\mathrm{d}u = 0$. Therefore, in null coordinates,

$$\mathrm{d}r = \frac{\partial r}{\partial v}\mathrm{d}v = \frac{\mathscr{A}}{2}\left[\cosh v - \cosh\frac{v-u}{2}\right]\mathrm{d}v = \mathscr{A}[\cosh(\eta+\chi) - \cosh\chi]\mathrm{d}\chi,$$

so that the angular components of the AKV are

$$\zeta^A = Y^A - \mathscr{D}^A f\int_r^\infty \mathrm{d}\chi\,\operatorname{csch}^2\chi + \mathscr{O}\big(1/r^3\big) = Y^A - 2\bigg(\frac{1}{1 - e^{-2\chi(r)}} - 1\bigg)\mathscr{D}^A f + \mathscr{O}\big(1/r^3\big).$$

The coefficient $C(r)$ of the $\mathscr{D}^A f$ term vanishes faster than any polynomial of $\mathscr{O}(1/r^n)$; i.e., $\lim_{r\to\infty} C(r)\mathscr{O}(r^n) = 0 \iff C(r) = \mathscr{O}(e^{-r})$. This essential singularity means that, with respect to the boundary conditions (5.5), the supertranslation AKV $\boldsymbol{\zeta}_T$ is trivial on $\mathscr{I}^+$, and the usual group of supertranslation AKVs (5.6) is not recovered.

6 Results and Discussion

This project sought to investigate the asymptotic structure and symmetries of a class of FLRW spacetimes, extending the analysis usually applied in the literature to asymptotically Minkowski spacetime. The dust-filled open FLRW universe was shown to be asymptotic to the Milne universe, suggesting that it may be asymptotically flat to the future, in an appropriate sense. Toward the goal of making this sense more concrete, the Bondi–Sachs flatness of the open FLRW universe was compared to spatially flat FLRW universes, along with the rate of decay of null geodesic deviation, as approximate indicators of asymptotic behaviour.

We determined that no FLRW universe was asymptotically flat in the sense of Bondi–Sachs, and found the degrees to which the studied open and spatially flat FLRW universes failed the criterion. The spatially hyperbolic (open) FLRW universe was found to be 'flatter' than any spatially flat FLRW model governed by a linear equation of state: its metric components deviate from $\mathcal{O}(1)$ along outgoing null geodesics as $\mathcal{O}(\ln r)$, versus $\mathcal{O}\big(r^{q/2}\big), q > 0$ in the spatially flat case. The physical significance of this preliminary result is subtle and warranted further study.

The asymptotic deviations of null geodesics in such FLRW spacetimes were computed to leading order and compared. We found the spatially flat FLRW universes to be characterised by a falloff rate of $\mathcal{O}(1/t^2)$, and the dust-filled spatially hyperbolic FLRW universe by a faster falloff of $\mathcal{O}(1/t^3)$. To a first approximation, the latter compares to the $\mathcal{O}(1/t^3)$ falloff in the Schwarzschild geometry (which is Bondi–Sachs asymptotically flat), despite relevant differences in the type of curvature. While the dust-filled open FLRW universe is purely Ricci curved, the curvature of the vacuum Schwarzschild geometry is purely Weyl (as are the asymptotically flat spacetimes generally considered in the literature, taken sufficiently far from sources). An observer only able to measure the evolution cross section of a beam of light would be able to distinguish the Schwarzschild geometry from the spatially hyperbolic FLRW universe by measuring shearing of the beam or taking into account higher-order terms. However, to first approximation, the rates of decay of null geodesic deviation are equivalent. This further motivates the categorisation of the open FLRW universe as 'asymptotically flat' in a new sense.

The asymptotic region of spacetime was defined more precisely in the Bondi–Sachs formalism, and the topologies of past and future null infinity were established for the class of FLRW universes in question. We compactified the open FLRW universe, and showed that the FLRW universes possess future null infinities $\mathscr{I}^+$ of the same topology as Minkowski space. This lead to the inspection of the asymptotic symmetries of $\mathscr{I}^+$.

Based off the existence of $\mathscr{I}^+$, Kehagias et al. [5] argue non-rigorously that the asymptotic symmetry group of decelerating spatially flat FLRW universes is identically the BMS group of asymptotically flat spacetime. We explicitly computed the asymptotic Killing vectors of these universes, and of the open FLRW universe, in accordance with the more explicit formulation of asymptotic symmetries in terms of the asymptotic Killing equation, as described in [3, 21, 26]. We found that the AKVs of decelerating spatially flat FLRW universes differ to the asymptotically flat case by a term of order $\mathcal{O}\big(1/r^h\big)$ in the angular components, where $0 < h < 1$ depends on the matter's equation of state. This results in a AKV whose norm becomes infinite as $r \to \infty$, revealing that the group of AKVs at $\mathscr{I}^+$ is inequivalent to the asymptotically flat case. This does not necessarily mean that decelerating spatially flat FLRW universes do not admit the exact BMS group of asymptotic symmetries; the relationship between AKVs and physically relevant asymptotic symmetries is subtle, and is the natural next step in this inquiry. It is conceivable that a different choice of boundary conditions δg_{ab} would yield a group of AKVs which leads to an asymptotic symmetry group equivalent to the BMS group, consistent with Kehagias et al.

Contrary to the spatially flat case, the dust-filled spatially hyperbolic FLRW universe was found to admit a supertranslation AKV whose angular components vanished as $\mathcal{O}(e^{-r})$. This means that the action of supertranslations is trivial on $\mathscr{I}^+$ in the open FLRW universe, strongly suggesting that the BMS group does not arise in a physically significant way. For this reason, if a notion of asymptotic 'future' flatness were to be defined which included the spatially hyperbolic open FLRW universe as an example (and encapsulated its 'flat-like' properties discussed earlier), then it would be qualitatively different from Bondi–Sachs asymptotic flatness—at least to the degree that the BMS group does not arise in such a spacetime. This may indicate a differing nature of gravitational memory in such universes. However, the link between AKVs and asymptotic symmetries is required to make this rigorous.

Further research should include: (i) a proper analysis of the relationship between asymptotic Killing vectors, asymptotic symmetries, and their associated physical charges; and (ii) further investigation into the physical motivation behind the boundary conditions used to define the asymptotic Killing equation, and their affect on the resulting group of AKVs. The gravitational memory effect is directly linked to the charges associated with asymptotic symmetries, and these depend intimately on the prescription of boundary conditions. The theme of the open FLRW universe being regarded as 'asymptotically flat' should also be made more precise—concretising this notion was one of the original intentions behind this investigation. Looking forward, this path of research may ultimately lead to insight into the nature of gravitational memory in more general spacetimes.

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A Geodesic Deviation in Schwarzschild Spacetime

The Schwarzschild geometry in Cartesian isotropic coordinates $(t, x, y, z) =: (t, x^i)$ is

$$\mathrm{d}s^2 = -\left(\frac{1 - m/r}{1 + m/r}\right)^2 c^2 \mathrm{d}t^2 + \left(1 + \frac{m}{r}\right)^4 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

where $m := r_s/4 = GM/2$ in terms of the Schwarzschild radius r_s or mass M of the black hole. We are interested in the order with which the geodesic deviation

$$\ddot{\boldsymbol{\xi}} := \frac{\mathrm{D}^2 \xi^\mu}{\mathrm{d}\lambda^2} = R^\mu{}_{\rho\sigma\nu} P^\rho P^\sigma \xi^\nu \tag{A.1}$$

of null geodesics $x^\mu(\lambda)$ with tangent vector P^μ depends on $r := \sqrt{\delta_{ij} x^i x^j}$. Taking advantage of spherical symmetry, we may assume that $\boldsymbol{P}$ is in the zt-plane, and that the separation vector $\boldsymbol{\xi}$ connecting nearby geodesics is proportional to ∂_x. Requiring $\|P\| = 0$ implies

$$\boldsymbol{P} \propto \frac{r + m}{r - m} \partial_t + \frac{r^2}{(m + r)^2} \partial_z,$$

and with $\boldsymbol{\xi} \propto \partial_x$, the geodesic deviation equation (A.1) reduces to a summation on two indices only,

$$\frac{\mathrm{D}^2 \xi^x}{\mathrm{d}\lambda^2} = R^x{}_{\rho\sigma x} P^\rho P^\sigma \xi^x \quad \text{(no sum on } x\text{)}.$$

(We expect $\ddot{\xi}^y = \ddot{\xi}^z = 0$ on the basis of spherical symmetry, and discard $\ddot{\xi}^z$ since we are interested in the area of a beam of area $A = (\xi^x)^2$.) The relevant components of the Riemann tensor with respect to the Cartesian isotropic coordinate frame are[1]

$$R^x{}_{ttx} = \frac{2(3\cos^2\phi \sin^2\phi - 1)(m - r)^2 m r^3}{(m + r)^8}, \qquad R^x{}_{zzx} = -\frac{2(3\sin^2\phi \sin^2\theta - 1)m}{(m + r)^2 r},$$

where $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ and $z = \cos\theta$. Evaluated at a distance r along the x-axis $(r, \theta, \phi) = (r, \pi/2, 0)$, the geodesic deviation of light rays in the radial component is

$$\ddot{\boldsymbol{\xi}} = \frac{6 m r^5}{(m + r)^8} \partial_x = \mathcal{O}\big(1/r^3\big). \tag{A.2}$$

As $r \to \infty$ along outgoing null rays, (A.2) is equivalently $\mathcal{O}(1/t^3)$, since $ct/r \to 1$.

[1]Computed with *SageMath*, an open-source mathematics software system under the general public license with differential geometry capabilities; see http://www.sagemath.org/.

B AKVs of Asymptotically Minkowski Spacetime

Here, the asymptotic Killing vectors (AKVs) satisfying

$$\mathscr{L}_{\boldsymbol{\zeta}} g_{ab} = \delta g_{ab} \tag{B.1}$$

of asymptotically Minkowski spacetime are derived, where δg_{ab} are required to preserve the Bondi gauge (see section 3.1),

$$g_{rr} = \delta g_{rr} = 0 \qquad g_{rA} = \delta g_{rA} = 0 \qquad \partial_r \det\big(g_{AB}/r^2\big) = g^{AB}\delta g_{AB} = 0$$

and to satisfy the boundary conditions

$$g_{uu} = -1 + \mathcal{O}(1/r), \quad g_{ur} = -1 + \mathcal{O}\big(1/r^2\big), \quad g_{uA} = \mathcal{O}\big(1/r^2\big), \quad g_{AB} = r^2 q_{AB} + \mathcal{O}(1), \tag{B.2}$$

as prescribed by BMS (see [3, §5.2.1]). Taking advantage of the Bondi gauge, the rr-component of the AKV equation (B.1) is $\mathscr{L}_{\boldsymbol{\zeta}} g_{rr} = 0$. Expanding the Lie derivative,

$$\mathscr{L}_{\boldsymbol{\zeta}} g_{rr} = g_{rr,\sigma}\zeta^\sigma + 2g_{\sigma r}\zeta^\sigma{}_{,r} = 2g_{ur}\zeta^u{}_{,r} = 0 \qquad \Longrightarrow \qquad \boxed{\zeta^u = f(u, x^A)}. \tag{B.3}$$

Similarly, the rA-components yield, using (B.3),

$$0 = \mathscr{L}_{\boldsymbol{\zeta}} g_{rA} = \big[r^2 q_{AB} + \mathcal{O}(1)\big]\zeta^B{}_{,r} + \big[-1 + \mathcal{O}\big(1/r^2\big)\big] f_{,A}$$

where $\mathscr{D}_A f = f_{,A}$ denotes the covariant derivative on the 2-sphere. Let n in $\zeta^B = \mathcal{O}(r^n)$ be the yet unknown falloff rate of ζ^B. Note that $\mathcal{O}(1/r^2)\zeta^B{}_{,r} = \mathcal{O}(r^{n-3})$, so that, solving for $\zeta^B{}_{,r}$ by contracting with q^{CA}, we have

$$q^{CA} q_{AB}\zeta^B{}_{,r} + \mathcal{O}\big(r^{n-3}\big) = \frac{1}{r^2}\mathscr{D}^C f + \mathcal{O}\big(1/r^4\big).$$

Upon integrating, with $Y^A = Y^A\big(u, x^A\big)$ as constants of integration,

$$\zeta^A + \mathcal{O}\big(r^{n-2}\big) = Y^A - \frac{1}{r}\mathscr{D}^A f + \mathcal{O}\big(1/r^3\big).$$

From $\zeta^A \equiv \mathcal{O}(r^n)$ and (B.4), $\mathcal{O}(r^n) = \mathcal{O}(1/r^3) + \mathcal{O}(r^{n-2})$, so $n = \max\{-3, n-2\} \implies n = -3$, so we may drop the $\mathcal{O}(r^{n-2})$ term in (B.4).

$$\boxed{\zeta^A = Y^A - \frac{1}{r}\mathscr{D}^A f + \mathcal{O}\big(1/r^3\big)} \tag{B.4}$$

Finally, the r-component of $\boldsymbol{\zeta}$ may be found by using the fact that, in the Bondi gauge, $\partial_r \det(g_{AB}/r^2) = 0$, which is equivalent to $g^{AB}\mathscr{L}_{\boldsymbol{\zeta}} g_{AB} = 0$. The AB-components of (B.1) are

$$\begin{aligned}\mathscr{L}_{\boldsymbol{\zeta}} g_{AB} &= g_{AB,\sigma}\zeta^\sigma + 2g_{\sigma(A}\zeta^\sigma{}_{,B)} \\ &= 2rq_{AB}\zeta^r + r^2 q_{AB,C}\zeta^C + 2r^2 q_{C(A}\zeta^C{}_{,B)} + \mathcal{O}(1).\end{aligned}$$

Note that $\zeta^u = f(u, x^A) = \mathcal{O}(1)$, and $g_{AB} = r^2 q_{AB} + \mathcal{O}(1) \iff g^{AB} = \frac{1}{r^2}q^{AB} + \mathcal{O}(1/r^4)$.

$$\begin{aligned}0 = g^{AB}\mathscr{L}_{\boldsymbol{\zeta}} g_{AB} &= \frac{1}{r^2}q^{AB}\mathscr{L}_{\boldsymbol{\zeta}} g_{AB} + \mathcal{O}(1/r^4) \\ &= \frac{2}{r}q^{AB}q_{AB}\zeta^r + q^{AB}q_{AB,C}\zeta^C + 2q^{AB}q_{CA}\zeta^C{}_{,B} + \mathcal{O}(1/r^4) \\ &= \frac{4}{r}\zeta^r + q^{AB}q_{AB,C}\zeta^C + 2\zeta^A{}_{,A} + \mathcal{O}(1/r^4).\end{aligned} \tag{B.5}$$

The middle two terms in (B.5) may be rewritten in terms of the covariant derivative on the 2-sphere, using

$$q^{AB}\left(q_{AB,C}\zeta^C + 2q_{C(A}\zeta^C{}_{,B)}\right) = q^{AB}\mathscr{L}^{(q)}_{\boldsymbol{\zeta}} q_{AB} = 2q^{AB}\mathscr{D}_A\zeta_B = 2\mathscr{D}_A\zeta^A$$

where the Lie derivative is *on the 2-sphere only*, and the second equality is Killing's equation on the 2-sphere. Finally, substituting (B.4), we obtain

$$\boxed{\zeta^r = -\frac{r}{2}\mathscr{D}_A Y^A + \frac{1}{2}\mathscr{D}^2 f + \mathcal{O}(1/r^3)}.$$

Hence, the general unconstrained AKV is

$$\boldsymbol{\zeta} = f\partial_u + \left[-\frac{r}{2}\mathscr{D}_A Y^A + \frac{1}{2}\mathscr{D}^2 f + \mathcal{O}(1/r^3)\right]\partial_r + \left[Y^A - \frac{1}{r}\mathscr{D}^A f + \mathcal{O}(1/r^3)\right]\partial_A.$$

Now we further constrain the forms of $f(u, x^A)$ and $Y^A(u, x^C)$ by enforcing the boundary conditions (B.2). The ur-component of (B.1) yields

$$\begin{aligned}\mathcal{O}(1/r^2) = \mathscr{L}_{\boldsymbol{\zeta}} g_{ur} &= g_{\sigma r}\zeta^\sigma{}_{,u} + g_{u\sigma}\zeta^\sigma{}_{,r} \\ &= g_{ur}\zeta^u{}_{,u} + g_{ur}\zeta^r{}_{,r} + g_{uA}\zeta^A{}_{,r} \\ &= -f_{,u} - \frac{1}{2}\mathscr{D}_A Y^A + \mathcal{O}(1/r^2)\end{aligned}$$

This implies $f_{,u} = \frac{1}{2}\mathscr{D}_A Y^A + \mathcal{O}(1/r^2)$, or (remembering that $f = f(u, x^A)$) is independent of r,

$$f(u, x^A) = T(x^A) + \frac{u}{2}\mathscr{D}_A Y^A + \mathcal{O}(1/r).$$

Finally, the uA-components lead to

$$
\begin{aligned}
\mathcal{O}(1) = \mathscr{L}_{\boldsymbol{\zeta}} g_{uA} &= \underbrace{g_{uA,\sigma}\zeta^\sigma}_{\leq \mathcal{O}(1)} + g_{\sigma A}\zeta^\sigma{}_{,u} + g_{u\sigma}\zeta^\sigma{}_{,A} \\
&= \cancelto{\mathcal{O}(1)}{g_{uA}\zeta^u{}_{,u}} + g_{BA}\zeta^B{}_{,u} + g_{uu}\underbrace{\zeta^u{}_{,A}}_{\mathcal{O}(1)} + g_{ur}\zeta^r{}_{,A} + \underbrace{g_{uB}}_{\mathcal{O}(1)}\zeta^B{}_{,A} + \mathcal{O}(1) \\
&= \big(r^2 q_{BA} + \mathcal{O}(r)\big)\left[Y^A{}_{,u} - \frac{1}{r} f_,{}^A{}_u\right] \\
&+ \big(-1 + \mathcal{O}\big(1/r^2\big)\big)\left[\frac{r}{2}\big(\mathscr{D}_B Y^B\big)_{,A} - \frac{1}{2}\big(\mathscr{D}^2 f\big)_{,A} + \mathcal{O}\big(1/r^2\big)\right] + \mathcal{O}(1) \\
&= r^2 q_{AB} Y^A{}_{,u} + \mathcal{O}(r).
\end{aligned}
$$

So we have $r^2 Y_{A,u} = \mathcal{O}(r)$, which implies $Y^A{}_{,u} = 0$, i.e., that $Y^A = Y^A\big(x^B\big)$ are functions of the 2-sphere.