Central limit theorem

Let $(X_1, ..., X_N)$ be independent and identically distributed random variables with mean μ and variance σ^2.

$$\boxed{\frac{1}{Z}\sum_{i=1}^{N} X_i \xrightarrow{d} \mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right)}$$

In other words, their mean converges in distribution to a normal distribution with standard deviation $\frac{\sigma}{\sqrt{N}}$.

Proof. We are interested in the distribution of the mean $I := \frac{1}{N}\sum_{i=1}^{N} X_i$, regarded as a random variable itself. Define standardised variables $Z_i = \frac{X_i - \mu}{\sigma}$ so that $\langle Z_i \rangle = 0$ and $\langle Z_i^2 \rangle = 1$. Define $\xi_N := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} Z_i$. We will show that $\xi_N \longrightarrow \mathcal{N}(0, 1)$ as $N \longrightarrow \infty$ by showing that ξ has the same moment-generating function as the standard normal distribution.

$$\begin{aligned}
M_{\xi_N}(t) &= \left\langle \exp \frac{t}{\sqrt{N}} \sum_i Z_i \right\rangle \\
&= \left\langle \prod_i \exp \frac{t}{\sqrt{N}} Z_i \right\rangle \\
&= \prod_i \left\langle \exp \frac{t}{\sqrt{N}} Z_i \right\rangle \\
&= \left\langle \exp \frac{t}{\sqrt{N}} Z_1 \right\rangle^N \\
&= \left(1 + \frac{t}{\sqrt{N}}\langle Z_1 \rangle + \frac{t^2}{2N}\langle Z_1^2 \rangle + \cdots\right)^N \\
&= \left(1 + \frac{t^2}{2N} + \mathcal{O}\left(N^{-\frac{3}{2}}\right)\right)^N
\end{aligned}$$

Therefore, in the limit, $\lim_{N\to\infty} M_{\xi_N}(t) = \exp\left(\frac{t^2}{2}\right)$, which is the moment-generating function for $\mathcal{N}(0, 1)$.

Finally, note that

$$\xi = \frac{1}{\sqrt{N}}\sum_i Z_i = \frac{1}{\sigma\sqrt{N}}\sum_i (X_i - \mu) = \frac{\sqrt{N}}{\sigma}\left(\frac{1}{N}\sum_i X_i\right) - \frac{N\mu}{\sigma\sqrt{N}} = \frac{\sqrt{N}}{\sigma}(I - \mu)$$

which implies that $I = \frac{\sigma}{\sqrt{N}}\xi + \mu \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right)$.