(Anti)centralizers

Let G be an associative algebra (for example, a geometric algebra). For any invertible element $a \in G$, define the *(anti)commuting part* of $b \in G$ as

$$\mathrm{Z}_a^\pm(b) := \frac{1}{2}\big(b \pm aba^{-1}\big).$$

Lemma.

1. $ab = \pm ba \implies b = \mathrm{Z}_a^\pm(b)$
2. $ab = \pm ba \impliedby b = \mathrm{Z}_a^\pm(b)$ if $a^2 b = ba^2$

Proof. Assuming $ab = \pm ba$ then $\mathrm{Z}_a^\pm(b) = \frac{1}{2}\big(b + baa^{-1}\big) = b$. Going the other way, $a\mathrm{Z}_a^\pm(b) = \frac{1}{2}\big(ab \pm a^2ba^{-1}\big) = \pm\frac{1}{2}(\pm ab + ba) = \pm \mathrm{Z}_a^\pm(b)a$, but only provided $a^2ba^{-1} = ba$. ∎

From now on, assume the element a has a square a^2 which commutes with everything.

If a^2 is in the centre of G, define the *(anti)centralizer* of a given element $a \in G$ to be the vector space

$$\mathrm{Z}_a^\pm(G) := \big\{\mathrm{Z}_a^\pm(b) \mid b \in G\big\} = \{b \in G \mid ab = \pm ba\}$$

of elements which (anti)commute with a.

Lemma. The maps $\mathrm{Z}_a^\pm : G \to \mathrm{Z}_a^\pm(G)$ are projections so that $G = \mathrm{Z}_a^+(G) \oplus \mathrm{Z}_a^-(G)$.

Proof. The maps $\mathrm{Z}_a^\pm(b)$ are clearly linear in $b \in G$. They are idempotent since

$$\mathrm{Z}_a^\pm\big(\mathrm{Z}_a^\pm(b)\big) = \frac{1}{2}\big(\mathrm{Z}_a^\pm(b) \pm a\mathrm{Z}_a^\pm(b)a^{-1}\big) = \frac{1}{4}\big(b \pm 2aba^{-1} \pm a^2ba^{-2}\big) = \frac{1}{2}\big(b \pm aba^{-1}\big) = \mathrm{Z}_a^\pm(b)$$

and are hence projections. Finally, since $\mathrm{Z}_a^+(b) + \mathrm{Z}_a^-(b) = b$, any element is of the form $b = b^+ + b^-$ where $b^\pm \in \mathrm{Z}_a^\pm(G)$. ∎

Lemma. $G = \mathrm{Z}_a^+(G) \oplus \mathrm{Z}_a^-(G)$ forms a $\mathbb{Z}_2$-grading: elements multiply under the geometric product according to the multiplication table:

	$\mathrm{Z}_a^+(G)$	$\mathrm{Z}_a^-(G)$
$\mathrm{Z}_a^+(G)$	$\mathrm{Z}_a^+(G)$	$\mathrm{Z}_a^-(G)$
$\mathrm{Z}_a^-(G)$	$\mathrm{Z}_a^-(G)$	$\mathrm{Z}_a^+(G)$

Proof. Let $b \in \mathrm{Z}_a^+(G)$ and $c \in \mathrm{Z}_a^\pm(G)$. Then $abc = bac = \pm bca$ so $bc \in \mathrm{Z}_a^\pm(G)$. This shows the first row/column of the table. Now if $b \in \mathrm{Z}_a^-(G)$ with c the same, we have $abc = -bac = \mp bca$ so $bc \in \mathrm{Z}_a^\mp(G)$. This completes the table. ∎