Classification of blades in CGA

Consider an arbitrary k-blade in a conformal geometric algebra $Cl(n+1,1) \cong \mathbb{R}^{n+2}$, assuming a basis $\{o, e_1, \dots, e_n, \infty\}$ where $o^2 = \infty^2 = 0$ and $o \cdot \infty = -1$.

Result. Any k-blade X is of exactly one of the forms

$$X = \begin{cases} \mathsf{T}_p[\boldsymbol{E}] \\ \mathsf{T}_p[\boldsymbol{E} \wedge \infty] = \boldsymbol{E}\infty \\ \mathsf{T}_p[o \wedge \boldsymbol{E} \wedge \infty] \\ \mathsf{T}_p\big[(o \pm \tfrac{1}{2}r^2\infty) \wedge \boldsymbol{E}\big] \end{cases}$$

where $\boldsymbol{E}$ is a Euclidean blade in the base space $Cl(n)$, $\boldsymbol{p} \in \mathbb{R}^n$ is a Euclidean position vector and $r \geq 0$ is a radius. The translation operator $\mathsf{T}_p[X] \equiv \mathsf{T}_p \; X \; \widetilde{\mathsf{T}_p}$ has the form

$$\mathsf{T}_p = \exp\big(\tfrac{1}{2}\infty\boldsymbol{p}\big) = 1 + \tfrac{1}{2}\infty\boldsymbol{p}$$

and satisfies $\mathsf{T}_p[\mathrm{up}(\boldsymbol{x})] = \mathrm{up}(\boldsymbol{x}+\boldsymbol{p})$ where $\mathrm{up}(\boldsymbol{x}) \coloneqq o + \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^2\infty$ and $\mathsf{T}_p[\infty] = \infty$.

Proof. We may write $X = u_1 \wedge \cdots \wedge u_k$ for linearly independent vectors u_i. Recall that the vectors can each be written in one of the following forms

$$u_i \propto \begin{cases} o + \boldsymbol{p} + \frac{1}{2}(\boldsymbol{p}^2 \pm r^2)\infty \\ \hat{\boldsymbol{n}} + d\infty \\ \infty \end{cases} \tag{1}$$

where $\propto$ denotes a nonzero scalar factor.

1. **Case** $X \wedge \infty = 0$. We have

$$X \propto u_1 \wedge \cdots \wedge u_{k-1} \wedge \infty$$

where, because of the ∞ factor, each u_i is of the form

$$u_i \propto \begin{cases} o + \boldsymbol{p} \\ \hat{\boldsymbol{n}} \end{cases}$$

obtained by rejecting the ∞ components of the forms in eq. 1.

   1. **Case** $X \lfloor \infty = 0$. This implies $u_i \lfloor \infty = 0$ for all u_i, so the only permissible form for each is $u_i \propto \hat{\boldsymbol{n}}$. In other words,

   $$X \propto \boldsymbol{E} \wedge \infty = \boldsymbol{E}\infty$$

   where $\boldsymbol{E} \coloneqq \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_{k-1}$ is a blade in the base space, $\boldsymbol{v}_i \in \mathrm{span}\{e_1, \dots, e_n\} \cong \mathbb{R}^n$.

   2. **Case** $X \lfloor \infty \neq 0$. This implies there is at least one u_i for which $u_i \lfloor \infty \neq 0$, meaning $u_i \propto o + \boldsymbol{p}_i$. Without loss of generality, say $u_1 \propto o + \boldsymbol{p}$. We can add a multiple of ∞ to u_1 to get

   $$X \propto (o + \boldsymbol{p} + \tfrac{1}{2}\boldsymbol{p}^2\infty) \wedge u_2 \wedge \cdots \wedge u_{k-1} \wedge \infty$$

   since this is annihilated by the ∞ factor in X. The first factor is now the conformal point $\mathrm{up}(\boldsymbol{p})$. Using the translation operator T_p,

   $$\begin{aligned} X &\propto \mathsf{T}_p[o] \wedge u_2 \wedge \cdots \wedge u_{k-1} \wedge \infty \\ &= \mathsf{T}_p[o \wedge u_2' \wedge \cdots \wedge u_{k-1}' \wedge \infty] \end{aligned}$$

   where $u_i' \coloneqq \mathsf{T}_{-p}[u_i]$ are translated vectors of the general form in eq. 1. However, because of factors of o and ∞, we may reject those components in each of the u_i' so they become purely Euclidean vectors, $u_i' \in \mathrm{span}\{e_1, \dots, e_n\}$. The final form is then

   $$X \propto \mathsf{T}_p[o \wedge \boldsymbol{E} \wedge \infty]$$

   where $\boldsymbol{E}$ is a $(k-2)$-blade in the base space.

2. **Case** $X \wedge \infty \neq 0$. We have

$$X \propto u_1 \wedge \cdots \wedge u_k$$

where none of the u_i are scalar multiples of ∞, so are of the form:

$$u_i \propto \begin{cases} o + \boldsymbol{p} + \frac{1}{2}(\boldsymbol{p}^2 \pm r^2)\infty \\ \hat{\boldsymbol{n}} + d\infty \end{cases}$$

   1. **Case** $X \lfloor \infty = 0$. This implies $u_i \lfloor \infty = 0$ for each i meaning all the vectors u_i are of the form $\boldsymbol{n}_i + d_i\infty$ and we have

   $$X \propto (\boldsymbol{n}_1 + d_1\infty) \wedge \cdots \wedge (\boldsymbol{n}_k + d_k\infty)$$

   for $\boldsymbol{n}_i \in \mathrm{span}\{e_1, \dots, e_n\}$ and $d_i \in \mathbb{R}$. We may perform the Gram–Schmidt process on the factors of X so that $(\boldsymbol{n}_i + d_i\infty) \cdot (\boldsymbol{n}_j + d_j\infty) = \boldsymbol{n}_i \cdot \boldsymbol{n}_j = 0$ whenever $i \neq j$.

   Note that translating base space vectors gives

   $$\mathsf{T}_p[\boldsymbol{n}_i] = \boldsymbol{n}_i + (\boldsymbol{n}_i \cdot \boldsymbol{p})\infty$$

   so if we choose $\boldsymbol{p} \coloneqq \sum_{i=1}^k \frac{d_i}{\|\boldsymbol{n}_i\|^2}\boldsymbol{n}_i$ so that $\boldsymbol{n}_i \cdot \boldsymbol{p} = d_i$ (remember the $\boldsymbol{n}_i$ are mutually orthogonal) then we can rewrite all the factors as

   $$X \propto \mathsf{T}_p[\boldsymbol{n}_1] \wedge \cdots \wedge \mathsf{T}_p[\boldsymbol{n}_k] = \mathsf{T}_p[\boldsymbol{E}]$$

   where $\boldsymbol{E} = \boldsymbol{n}_1 \wedge \cdots \wedge \boldsymbol{n}_k$ is a k-blade in the base space.

   2. **Case** $X \lfloor \infty \neq 0$. This implies there is at least one u_i for which $u_i \lfloor \infty \neq 0$. Assume it is the first one. Then since it is only form allowed by eq. 2 we must have $u_1 \propto o + \boldsymbol{q} + \frac{1}{2}(\boldsymbol{q}^2 \pm r^2)\infty$. Furthermore, we may take $u_2, \dots, u_k$ to be of the form $u_i \propto \boldsymbol{n}_i + d_i\infty$, since we can always subtract an appropriate multiple of u_1 in order to eliminate any o component. Even further, we may assume that $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = 0$ by performing Gram–Schmidt on $u_2, \dots, u_k$ as in the previous case. Now we have:

   $$X \propto (o + \boldsymbol{q} + \alpha\infty) \wedge (\boldsymbol{n}_2 + d_2\infty) \wedge \cdots \wedge (\boldsymbol{n}_k + d_k\infty)$$

   Similarly to eq. 3, the trailing factors above may be written as $\mathsf{T}_p[\boldsymbol{E}]$ with $\boldsymbol{E} = \boldsymbol{n}_2 \wedge \cdots \wedge \boldsymbol{n}_k$ by choosing the translation vector $\boldsymbol{p} \coloneqq \sum_{i=2}^k \frac{d_i}{\|\boldsymbol{n}_i\|^2}\boldsymbol{n}_i$. Putting everything inside this translation operator:

   $$X \propto \mathsf{T}_p[(o + \boldsymbol{q}' + \alpha'\infty) \wedge \boldsymbol{E}]$$

   It does not matter what $\boldsymbol{q}'$ and α' are exactly — we know it is of this form. Furthermore, we may take $\boldsymbol{q}'$ to be perpendicular to $\boldsymbol{E}$, since any parallel component is annihilated by the wedge product. Now, if $\boldsymbol{q}' \cdot \boldsymbol{E} = 0$ then $\mathsf{T}_{q'}[\boldsymbol{E}] = \boldsymbol{E}$ (see the translation law for blades) which means we can translate again to get rid of $\boldsymbol{q}'$.

   $$X \propto \mathsf{T}_{p-q'}[(o + \alpha''\infty) \wedge \boldsymbol{E}]$$

   Finally, relabelling $\boldsymbol{p} - \boldsymbol{q}' \mapsto \boldsymbol{p}$ and $\alpha \mapsto \pm\frac{1}{2}r^2$ we have the final form

   $$X = \mathsf{T}_p\big[(o \pm \tfrac{1}{2}r^2\infty) \wedge \boldsymbol{E}\big]$$

   where any constants of proportionality can be absorbed into $\boldsymbol{E}$ to make this an exact equality. ∎

Table 1: Summary of cases

	$X \wedge \infty = 0$	$X \wedge \infty \neq 0$
$X \lfloor \infty = 0$	$\boldsymbol{E}\infty$	$\mathsf{T}_p[\boldsymbol{E}]$
$X \lfloor \infty \neq 0$	$\mathsf{T}_p[o \wedge \boldsymbol{E} \wedge \infty]$	$\mathsf{T}_p\big[(o \pm \frac{1}{2}r^2\infty) \wedge \boldsymbol{E}\big]$