

Contractions between direct CGA objects

Suppose A and B are two directly represented objects in CGA, where A is of a higher grade than B . For instance, A might be a sphere and B a point or circle, etc. Consider their inner product:

$$C := A \cdot B = A \lfloor B$$

How can we characterise C geometrically in terms of A and B ?

We can deduce immediately that:

- $\text{gr}(C) = \text{gr}(A) - \text{gr}(B)$
- C directly represents a subset of A
From $\text{span}(C) \subseteq \text{span}(A)$ it follows that we can factorise $A = C \wedge Z$ for some Z . Therefore, $\text{up}(x) \wedge C = 0 \implies \text{up}(x) \wedge A = 0$ so $\text{OPNS}(C) \subseteq \text{OPNS}(A)$.
- C dually represents a superset of B
This follows because $\text{up}(x) \wedge B = 0$ implies $C \lfloor \text{up}(x) = (A \lfloor B) \lfloor \text{up}(x) = A \lfloor (B \wedge \text{up}(x)) = 0$ (double contraction identity) so $\text{OPNS}(B) \subseteq \text{IPNS}(C)$.
- C is invariant under reflection in A or B
First observe that $\text{span}(C) = \text{span}(A \lfloor B) \subseteq \text{span}(A)$ and also $\text{span}(C) \cap \text{span}(B) = \emptyset$. These imply that $CA = C \rfloor A$ and $CB = C \wedge B$ are both blades. The reverse of a blade is itself, up to a sign. This gives $CA \propto AC$ and $CB \propto BC$, or in other words $ACA \propto C$ and $BCB \propto C$. Geometrically, this means C is invariant under reflection in both A and B . Specifically, both $\text{OPNS}(C)$ and $\text{IPNS}(C)$ are transformed into themselves on being reflected in a mirror represented by A or B .

In the figures below, the grey object is the OPNS of C , and the yellow object is the IPNS. Dashed lines denote a round with an imaginary radius.

<p>A : 4-round B : 1-tangent $A \wedge B = 0$: incident $A \cdot B$: 3-tangent</p>		<p>A : 4-round B : 1-tangent $A \wedge B \neq 0$: not incident $A \cdot B$: imag. 3-round</p>	
<p>A : 4-round B : 2-tangent $A \wedge B = 0$: incident $A \cdot B$: 2-tangent</p>		<p>A : 4-round B : 2-tangent $A \wedge B \neq 0$: not incident $A \cdot B$: imag. 2-round</p>	
<p>A : 4-round B : 3-tangent $A \wedge B = 0$: incident $A \cdot B$: 1-tangent</p>		<p>A : 4-round B : 3-tangent $A \wedge B \neq 0$: not incident $A \cdot B$: imag.</p>	
<p>A : 4-round B : 2-round $A \cdot B$: 2-round</p>		<p>A : 4-round B : 3-round $A \cdot B$: 1-round</p>	
<p>A = 3-round B = 1-round $A \cdot B$ = 2-tangent</p>		<p>A = 3-round B = 1-round $A \cdot B$ = 2-round</p>	