Drills and exercises in CGA

Showing that the translation rotor translates points

$$
\begin{aligned}
n &:= e + \overline{e} \\
\overline{n} &:= e - \overline{e} \\
F(x) &:= \frac{1}{2}(x^2 n + 2x - \overline{n})
\end{aligned}
$$

Let $E = n \wedge \overline{n} = (e + \overline{e}) \wedge (e - \overline{e}) = 2\overline{e}e$. Note

$$E n = 2\overline{e}e(e + \overline{e}) = 2\overline{e}e^2 - 2e\overline{e}^2 = 2(\overline{e} + e) = 2n \tag{1}$$

$$E\overline{n} = 2\overline{e}e(e - \overline{e}) = 2\overline{e}e^2 + 2e\overline{e}^2 = 2(\overline{e} - e) = -2\overline{n} \tag{2}$$

Show that $R = \exp\left(-\frac{1}{2}nu\right)$ is a translation rotor.

$$
\begin{aligned}
RF(x)\tilde{R} &= \left(1 + \frac{1}{2}nu\right)\frac{1}{2}(x^2 n + 2x - \overline{n})\left(1 + \frac{1}{2}un\right) \\
&= \frac{1}{2}\left(x^2 n + 2x - \overline{n} + \frac{1}{2}x^2 \cancel{nun} + nux - \frac{1}{2}nu\overline{n}\right)\left(1 + \frac{1}{2}un\right) \\
&= \frac{1}{2}\left(\left[x^2 + ux\right]n + 2x - \frac{1}{2}nu\overline{n} - \overline{n}\right)\left(1 + \frac{1}{2}un\right) \\
&= \frac{1}{2}\Bigg(\left[x^2 + ux\right]n + 2x + \frac{1}{2}un\overline{n} - \overline{n} \\
&\qquad \left[x^2 + ux\right]\frac{1}{2}\cancel{nun} + xun - \frac{1}{4}nu\overline{n}un - \frac{1}{2}\overline{n}un\Bigg) \\
&= \frac{1}{2}\left(\left[x^2 + ux + xu\right]n + 2x + \frac{1}{2}u(n\overline{n} + \overline{n}n) + \frac{1}{4}u^2 n\overline{n}n - \overline{n}\right) \\
&= \frac{1}{2}\left(\left[x^2 + ux + xu\right]n + 2x + u(n \cdot \overline{n}) + \frac{1}{4}u^2(n \cdot \overline{n} + E)n - \overline{n}\right) \\
&= \frac{1}{2}\left(\left[x^2 + ux + xu\right]n + 2x + 2u + \frac{1}{4}u^2(2 + 2)n - \overline{n}\right) \\
&= \frac{1}{2}(\left[x^2 + ux + xu\right]n + 2(x + u) + u^2 n - \overline{n}) \\
&= \frac{1}{2}(\left[x^2 + ux + xu + u^2\right]n + 2(x + u) - \overline{n}) \\
&= \frac{1}{2}([x + u]^2 n + 2[x + u] - \overline{n}) \\
&= F(x + u)
\end{aligned}
$$

The effect of T_p on Euclidean vectors

Let $\mathsf{T}_p := \exp\left(\frac{1}{2}\infty p\right)$ be the translation rotor and let $R[X] = RX\tilde{R}$ denote rotor application.

Let $p, u \in \mathbb{R}^n$.

$$
\begin{aligned}
T_p[u] &= \left(1 + \frac{1}{2}\infty p\right)u\left(1 + \frac{1}{2}p\infty\right) \\
&= u + \frac{1}{2}\infty pu + \frac{1}{2}up\infty + \frac{1}{2}\infty pup\infty \\
&= u + \frac{1}{2}(pu + up)\infty - \frac{1}{2}\cancel{\infty^2}pup \\
&= u + (u \cdot p)\infty
\end{aligned}
$$