

# Outer and Inner Product Null Spaces for CGA Blades

See also [summary of these results](#).

Using the [classification of blades in conformal geometric algebra](#),

$$X = \begin{cases} \mathbf{E}\infty & \text{direction} \\ \mathbb{T}_p[\mathbf{E}] & \text{dual flat} \\ \mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty] & \text{flat} \\ \mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] & \text{round} \end{cases}$$

we can easily characterise the inner and outer product null spaces

$$\begin{aligned} \text{ipns}(X) &:= \{x \in Q \mid x \rfloor X = 0\} \\ \text{opns}(X) &:= \{x \in Q \mid x \wedge X = 0\} \end{aligned}$$

of an arbitrary blade  $X$  where  $Q := \{u \in \mathbb{R}^{n+2} \mid u^2 = 0\} = \{\text{up}(x) \mid x \in \mathbb{R}^n\} \cup \{\infty\}$ .

## OPNS

- **Directions,  $\mathbf{E}\infty$ :**

Assuming  $\text{up}(x) \wedge (\mathbf{E}\infty) = 0$  implies

$$(\sigma + x + \frac{1}{2}x^2\infty) \wedge \mathbf{E} \wedge \infty = (\sigma + x) \wedge \mathbf{E} = 0$$

which is impossible because  $\sigma \wedge \mathbf{E}$  is always nonzero. Additionally,  $\infty \wedge \mathbf{E}\infty = 0$ .

Therefore  $\text{opns}(\mathbf{E}\infty) = \{\infty\}$ .

- **Dual flats,  $\mathbb{T}_p[\mathbf{E}]$ :**

Assuming  $\text{up}(x) \wedge \mathbb{T}_p[\mathbf{E}] = 0$  implies

$$\mathbb{T}_p[\text{up}(x - p) \wedge \mathbf{E}] = 0 \iff (\sigma + x - p + \frac{1}{2}(x - p)^2\infty) \wedge \mathbf{E} = 0$$

which is impossible because it includes the linearly independent term  $\sigma \wedge \mathbf{E} \neq 0$ .

Additionally,  $\infty \wedge \mathbb{T}_p[\mathbf{E}] = \mathbb{T}_p[\infty \mathbf{E}] \neq 0$ .

Therefore  $\text{opns}(\mathbb{T}_p[\mathbf{E}]) = \emptyset$ .

- **Flats,  $\mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty]$ :**

Assuming  $\text{up}(x) \wedge \mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty] = 0$  implies

$$\mathbb{T}_p[\text{up}(x - p) \wedge \sigma \wedge \mathbf{E} \wedge \infty] = 0 \iff (x - p) \wedge \sigma \wedge \mathbf{E} \wedge \infty = 0$$

which vanishes if and only if  $x = p + u$  for some vector  $u \in \text{span}(\mathbf{E}) \iff u \wedge \mathbf{E} = 0$ .

Additionally,  $\infty \wedge \mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty] = 0$ .

Therefore  $\text{opns}(\mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty]) = \{\text{up}(p + r) \mid r \wedge \mathbf{E} = 0\} \cup \{\infty\}$ .

- **Rounds,  $\mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}]$ :**

Assuming  $\text{up}(x) \wedge \mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] = 0$  implies

$$\begin{aligned} \mathbb{T}_p[\text{up}(x - p) \wedge (\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] &= 0 \iff \\ (\sigma + (x - p) + \frac{1}{2}(x - p)^2\infty) \wedge (\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E} & \\ = (\frac{1}{2}(x - p)^2 \mp \frac{1}{2}r^2)\infty \wedge \sigma \wedge \mathbf{E} + (x - p) \wedge (\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E} &= 0 \end{aligned}$$

which, since these two terms are linearly independent, means both must vanish

$$\|x - p\|^2 = \pm r^2 \quad \text{and} \quad (x - p) \wedge \mathbf{E} = 0$$

which, in the  $+$  case, describes a  $k$ -sphere within the flat  $x = p + u$  for  $u \in \text{span}(\mathbf{E})$

centred at  $p$  with radius  $r > 0$  (so  $u^2 = r^2$ ). Additionally,  $\infty \wedge \mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] \neq 0$ .

Therefore  $\text{opns}(\mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}]) = \{\text{up}(p + u) \mid u \wedge \mathbf{E} = 0, u^2 = \pm r^2\}$ .

## IPNS

- **Directions,  $\mathbf{E}\infty$ :**

Assuming  $\text{up}(x) \rfloor (\mathbf{E}\infty) = 0$  implies

$$\left(\sigma + x + \frac{x^2}{2}\infty\right) \rfloor (\mathbf{E}\infty) = (x \rfloor \mathbf{E})\infty + \mathbf{E}^*(\sigma \rfloor \infty) = (x \rfloor \mathbf{E}) \wedge \infty - \mathbf{E}^* = 0$$

which is impossible because the two terms are linearly independent. Additionally,  $\infty \rfloor \mathbf{E}\infty = 0$ .

Therefore  $\text{ipns}(\mathbf{E}\infty) = \{\infty\}$ .

- **Dual flats,  $\mathbb{T}_p[\mathbf{E}]$ :**

Assuming  $\text{up}(x) \rfloor \mathbb{T}_p[\mathbf{E}] = 0$  implies

$$\mathbb{T}_p[\text{up}(x - p) \rfloor \mathbf{E}] = 0 \iff (\sigma + x - p + \frac{1}{2}(x - p)^2\infty) \rfloor \mathbf{E} = (x - p) \rfloor \mathbf{E} = 0$$

which implies  $x = p + u$  for some vector  $u \in \text{span}(\mathbf{E})^\perp \iff u \rfloor \mathbf{E} = 0$ . Additionally,

$\infty \rfloor \mathbb{T}_p[\mathbf{E}] = \mathbb{T}_p[\infty \rfloor \mathbf{E}] = 0$ .

Therefore  $\text{ipns}(\mathbb{T}_p[\mathbf{E}]) = \{\text{up}(p + u) \mid u \rfloor \mathbf{E} = 0\} \cup \{\infty\}$ .

- **Flats,  $\mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty]$ :**

Assuming  $\text{up}(x) \rfloor \mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty] = 0$  implies

$$\begin{aligned} \mathbb{T}_p[\text{up}(x - p) \rfloor (\sigma \wedge \mathbf{E} \wedge \infty)] &= 0 \iff \\ (\sigma + (x - p) + \frac{1}{2}(x - p)^2\infty) \rfloor (\sigma \wedge \mathbf{E} \wedge \infty) & \\ = -\frac{1}{2}(x - p)^2\mathbf{E}\infty - \sigma \wedge ((x - p) \rfloor \mathbf{E}) \wedge \infty - \sigma \mathbf{E}^* \wedge (\sigma \rfloor \infty) &= 0 \end{aligned}$$

which is impossible because the terms are linearly independent and  $\sigma \mathbf{E}^* \neq 0$ .

Additionally,  $\infty \rfloor \mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty] = -\mathbb{T}_p[\mathbf{E}\infty] \neq 0$ .

Therefore  $\text{ipns}(\mathbb{T}_p[\sigma \wedge \mathbf{E} \wedge \infty]) = \emptyset$ .

- **Rounds,  $\mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}]$ :**

Assuming  $\text{up}(x) \rfloor \mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] = 0$  implies

$$\begin{aligned} \mathbb{T}_p[\text{up}(x - p) \rfloor ((\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E})] &= 0 \iff \\ ((\sigma + x - p + \frac{1}{2}(x - p)^2\infty) \rfloor (\sigma \pm \frac{1}{2}r^2\infty)) \wedge \mathbf{E} - (\sigma \pm \frac{1}{2}r^2\infty) \wedge (\text{up}(x - p) \rfloor \mathbf{E}) & \\ = -(\frac{1}{2}(x - p)^2 \pm \frac{1}{2}r^2) \wedge \mathbf{E} - (\sigma \pm \frac{1}{2}r^2\infty) \wedge ((x - p) \rfloor \mathbf{E}) &= 0 \end{aligned}$$

which, since these two terms are linearly independent, means both must vanish

$$\|x - p\|^2 = \mp r^2 \quad \text{and} \quad (x - p) \rfloor \mathbf{E} = 0$$

which, in the  $-$  case, describes a  $k$ -sphere within the flat  $x = p + u$  for  $u \in \text{span}(\mathbf{E})^\perp$

centred at  $p$  with radius  $r > 0$  (so  $u^2 = -r^2$ ). Additionally,  $\infty \rfloor \mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}] - \mathbb{T}_p[\mathbf{E}] \neq 0$ .

Therefore  $\text{ipns}(\mathbb{T}_p[(\sigma \pm \frac{1}{2}r^2\infty) \wedge \mathbf{E}]) = \{\text{up}(p + u) \mid u \rfloor \mathbf{E} = 0, u^2 = \mp r^2\}$ .