Reflections generated by blades in CGA

The idea is: understand rotors by classifying all reflections.

We have a nice classification of all blades given by

$$X = \begin{cases} \boldsymbol{E} n_\infty \\ \mathsf{T}_p[\boldsymbol{E}] \\ \mathsf{T}_p[n_0 \wedge \boldsymbol{E} \wedge n_\infty] \\ \mathsf{T}_p\left[\left(n_0 \pm \frac{\rho^2}{2} n_\infty\right) \wedge \boldsymbol{E}\right] \end{cases} \tag{1}$$

but what about a nice classification of all versors?

Any invertible blade A defines two transformations: a reflection of the vector space within A or across A. In a homogeneous setting, these are the same, since they differ only by an overall sign.

Not all orthogonal transformations are generated by reflections by blades. However, blades have geometrical interpretations via IPNS/OPNS, so they are a useful class of transformations to study.

Reflections as operators with nice properties

See to the note on projections and reflections. Let's define the operator

$$\mathrm{refl}(A, x) := A^\star x A^{-1} = u^{\perp A} - u^{\parallel A}$$

which reflects the multivector x within A. I.e., all the dimensions lying in the blade A are negated.

We have a simple composition rule:

$$\mathrm{refl}(A, \mathrm{refl}(B, x)) = A^\star B^\star x B^{-1} A^{-1} = (AB)^\star x (AB)^{-1} = \mathrm{refl}(AB, x)$$

In fact, $\mathrm{refl}(A, -)$ is an automorphism. In particular, $\mathrm{refl}(A, x^{-1}) = \mathrm{refl}(A, x)^{-1}$. If f is some automorphism, then $f(\mathrm{refl}(A, B)) = \mathrm{refl}(f(A), f(B))$.

Reflections and the embedding map

We have some nice properties: $\mathrm{up} \circ \mathrm{refl} = \mathrm{relf} \circ \mathrm{up}$

$$\begin{aligned} \mathrm{refl}(\boldsymbol{E}, \mathrm{up}(\boldsymbol{x})) &= \mathrm{refl}(\boldsymbol{E}, n_0) + \mathrm{refl}(\boldsymbol{E}, x) + \mathrm{refl}\left(\boldsymbol{E}, \tfrac{1}{2} x^2 n_\infty\right) \\ &= n_0 + \mathrm{refl}(\boldsymbol{E}, x) + \tfrac{1}{2}\boldsymbol{x}^2 n_\infty \\ &= \mathrm{up}(\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x})) \end{aligned}$$

$$\begin{aligned} \mathrm{refl}(n_0 \wedge n_\infty, \mathrm{up}(\boldsymbol{x})) &= \mathrm{refl}(n_0 \wedge n_\infty, n_0) + \boldsymbol{x} + \tfrac{1}{2} x^2 \ \mathrm{refl}(n_0 \wedge n_\infty, n_\infty) \\ &= -n_0 + \boldsymbol{x} - \tfrac{1}{2} x^2 n_\infty \\ &= -\mathrm{up}(-\boldsymbol{x}) \end{aligned}$$

Reflections of blades

Everything in eq. 1 except directions $\boldsymbol{E} \wedge n_\infty$ and zero-radius rounds $\mathsf{T}_p[n_0 \wedge \boldsymbol{E}]$ are invertible so generate reflections.

Reflection in a dual flat

$$\begin{aligned} \mathrm{refl}(\mathsf{T}_p[\boldsymbol{E}], \mathrm{up}(\boldsymbol{x})) &= \mathsf{T}_p[\mathrm{refl}(\boldsymbol{E}, \mathrm{up}(\boldsymbol{x} - \boldsymbol{p}))] \\ &= \mathsf{T}_p[\mathrm{up}(\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p}))] \\ &= \mathrm{up}(\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p}) + \boldsymbol{p}) \end{aligned}$$

Reflection in a flat

$$\begin{aligned} \mathrm{refl}(\mathsf{T}_p[n_0 \wedge \boldsymbol{E} \wedge n_\infty], \mathrm{up}(\boldsymbol{x})) &= \mathsf{T}_p[\mathrm{refl}(n_0 \wedge n_\infty \wedge \boldsymbol{E}^\star, \mathrm{up}(\boldsymbol{x} - \boldsymbol{p}))] \\ &= \mathsf{T}_p[\mathrm{refl}(n_0 \wedge n_\infty, \mathrm{refl}(\boldsymbol{E}^\star, \mathrm{up}(\boldsymbol{x} - \boldsymbol{p})))] \\ &= \mathsf{T}_p[\mathrm{refl}(n_0 \wedge n_\infty, \mathrm{up}(\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p})))] \\ &= \mathsf{T}_p[-\mathrm{up}(-\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p}))]) \\ &= -\mathrm{up}(\boldsymbol{p} - \mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p})))) \end{aligned}$$

Reflection in a round

We use the identities

$$\begin{aligned} \mathrm{refl}\left(n_0 \pm \tfrac{1}{2}\rho^2 n_\infty, n_0\right) &= \mp\tfrac{1}{2}\rho^2 n_\infty \\ \mathrm{refl}\left(n_0 \pm \tfrac{1}{2}\rho^2 n_\infty, n_\infty\right) &= \mp\frac{2}{\rho^2} n_0 \end{aligned}$$

we find

$$\begin{aligned} \mathrm{refl}\left(\mathsf{T}_p\left[\left(n_0 \pm \tfrac{1}{2}\rho^2 n_\infty\right) \wedge \boldsymbol{E}\right], \mathrm{up}(\boldsymbol{x})\right) &= \mathsf{T}_p\left[\mathrm{refl}\left(n_0 \pm \tfrac{1}{2}\rho^2 n_\infty, \mathrm{refl}(\boldsymbol{E}, \mathrm{up}(\boldsymbol{x} - \boldsymbol{p}))\right)\right] \\ &= \mathsf{T}_p\left[\mathrm{refl}\left(n_0 \pm \tfrac{1}{2}\rho^2 n_\infty, \mathrm{up}(\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p}))\right)\right] \\ &= \mathsf{T}_p\left[\mp\tfrac{1}{2}\rho^2 n_\infty + \mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p}) \mp \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{p})^2 \frac{2}{\rho^2} n_0\right] \\ &= \mp\left(\frac{\boldsymbol{x} - \boldsymbol{p}}{\rho}\right)^2 \mathsf{T}_p\left[n_0 \mp \frac{\rho^2}{\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p})} + \tfrac{1}{2}\left(\frac{\rho^2}{\boldsymbol{x} - \boldsymbol{p}}\right)^2 n_\infty\right] \\ &= \mp\left(\frac{\boldsymbol{x} - \boldsymbol{p}}{\rho}\right)^2 \mathsf{T}_p\left[\mathrm{up}\left(\mp\frac{\rho^2}{\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p})}\right)\right] \\ &= \mp\left(\frac{\boldsymbol{x} - \boldsymbol{p}}{\rho}\right)^2 \mathrm{up}\left(\boldsymbol{p} \mp \frac{\rho^2}{\mathrm{refl}(\boldsymbol{E}, \boldsymbol{x} - \boldsymbol{p})}\right) \end{aligned}$$

Fixed points of reflections

Suppose $\boldsymbol{x} \in \mathsf{OPNS}(X)$ so $\mathrm{up}(\boldsymbol{x}) \wedge X = 0$.

$$\mathrm{refl}(X, \mathrm{up}(\boldsymbol{x})) = A^\star\ \mathrm{up}(\boldsymbol{x}) A^{-1} = (A^\star \lfloor \mathrm{up}(\boldsymbol{x})) A^{-1} = (\mathrm{up}(\boldsymbol{x}) \rfloor A) A^{-1} = \mathrm{up}(\boldsymbol{x})$$

So $\mathsf{OPNS}(X)$ are fixed points of $\boldsymbol{x} \mapsto \mathrm{refl}(X, \mathrm{up}(\boldsymbol{x}))$.

Similarly, suppose $\boldsymbol{x} \in \mathsf{IPNS}(X)$ so $\mathrm{up}(\boldsymbol{x}) \rfloor X = 0$.

$$\mathrm{refl}(X, \mathrm{up}(\boldsymbol{x})) = A^\star\ \mathrm{up}(\boldsymbol{x}) A^{-1} = (A^\star \wedge \mathrm{up}(\boldsymbol{x})) A^{-1} = (\mathrm{up}(\boldsymbol{x}) \wedge A) A^{-1} = \mathrm{up}(\boldsymbol{x})$$

So $\mathsf{IPNS}(X)$ are fixed points of $\boldsymbol{x} \mapsto \mathrm{refl}(X, \mathrm{up}(\boldsymbol{x}))$.