Classification of 1-vectors in CGA

See also the classification of general blades.

Consider a nonzero vector X in a conformal geometric algebra $Cl(n+1,1) \cong \mathbb{R}^{n+2}$, assuming a basis $\{n_0, e_1, ..., e_n, n_\infty\}$ where $n_0^2 = n_\infty^2 = 0$ and $n_0 \cdot n_\infty = -1$.

Result. Any vector is of the form

$$X \propto \begin{cases} n_0 + \boldsymbol{p} + \frac{1}{2}(\boldsymbol{p}^2 \pm r^2)n_\infty \\ \hat{\boldsymbol{n}} + dn_\infty \\ n_\infty \end{cases} \tag{1}$$

where $\propto$ means equal up to a nonzero scale factor, $\boldsymbol{p} \in \mathbb{R}^n$ is a point, $r > 0$ is a radius, $\hat{\boldsymbol{n}}$ is a unit normal vector in $\mathbb{R}^n$ and $d \in \mathbb{R}$ is a distance.

Proof. A general nonzero vector is of the form

$$X = X_0 n_0 + \boldsymbol{p} + X_\infty n_\infty$$

where $X_0, X_\infty \in \mathbb{R}$ and $\boldsymbol{p} \in \operatorname{span}\{e_1, ..., e_n\} \cong \mathbb{R}^n$ is a vector in the base space.

1. **Case** $X_0 \neq 0$. The normalised form $\hat{X} = X/X_0$ is

$$\hat{X} = n_0 + \boldsymbol{p} + X_\infty n_\infty, \quad \hat{X}^2 = \mp r^2$$

   where $\boldsymbol{p}$ and X_∞ have been rescaled.

   1.1. **Case** $X_\infty < \frac{1}{2}\boldsymbol{p}^2$. There is some $r > 0$ such that $X_\infty = \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{2}r^2$.

$$\hat{X} = n_0 + \boldsymbol{p} + \frac{1}{2}(\boldsymbol{p}^2 - r^2)n_\infty, \quad \hat{X}^2 = r^2 > 0$$

   1.2. **Case** $X_\infty = \frac{1}{2}\boldsymbol{p}^2$.

$$\hat{X} = n_0 + \boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^2 n_\infty, \quad \hat{X}^2 = 0$$

   1.3. **Case** $X_\infty > \frac{1}{2}\boldsymbol{p}^2$. There is some $r > 0$ such that $X_\infty = \frac{1}{2}\boldsymbol{p}^2 + \frac{1}{2}r^2$.

$$\hat{X} = n_0 + \boldsymbol{p} + \frac{1}{2}(\boldsymbol{p}^2 + r^2)n_\infty, \quad \hat{X}^2 = r^2 < 0$$

2. **Case** $X_0 = 0$.

$$X = \boldsymbol{p} + X_\infty n_\infty, \quad X^2 = \boldsymbol{p}^2 > 0$$

   2.1. **Case** $\|\boldsymbol{p}\| \neq 0$. The normalised form $\hat{X} = X/\|\boldsymbol{p}\|$ is

$$\hat{X} = \hat{\boldsymbol{n}} + dn_\infty, \quad \hat{X}^2 = \hat{\boldsymbol{n}}^2 = 1 > 0$$

   where $\hat{\boldsymbol{n}} := \boldsymbol{p}/\|\boldsymbol{p}\|$ and $d := X_\infty/\|\boldsymbol{p}\|$.

   2.2. **Case** $\|\boldsymbol{p}\| = 0$. Over a nondegenerate base space, this implies $p = 0$ so that

$$\hat{X} = n_\infty, \quad \hat{X}^2 = 0$$

   where $\hat{X} = X/X_\infty$. ∎