Conformal Geometric Algebra

The conformal model in $\mathbb{R}^n$ is $Cl(n+1,1)$ with basis vectors

$$e_i^2 = 1 \text{ for } 1 \leq i \leq n \text{ and } e_\pm^2 = \pm 1$$

where we additionally may define

$$e_\infty \coloneqq \alpha(e_+ + e_-) \quad \text{and} \quad e_0 \coloneqq \beta(e_+ - e_-)$$

for $\alpha, \beta \in \mathbb{R}$ from which it follows $e_\infty^2 = e_0^2 = 0$ and $e_\infty \cdot e_0 = 2\alpha\beta$.

Inclusion map $Cl(n) \to Cl(n+1,1)$

Let $\iota : Cl(n) \to Cl(n+1,1)$ be the inclusion map induced by identifying sending $e_i \mapsto e_i$ in each algebra.

Detail. We can often forget about this map and simply understand that any element $a \in Cl(n)$ also belongs in $Cl(n+1,1)$. In this note, I'll still write it, but in a subtle shade, like $\iota(a) \in Cl(n+1,1)$.

Lifting map $\mathbb{R}^n \to Cl(n+1,1)$

Define the "upwards" lift

$$\begin{aligned} \text{up}(x) &\coloneqq \gamma e_0 + \iota(x) + \delta x^2 e_\infty \\ &= \iota(x) + (\alpha\delta x^2 + \beta\gamma)e_+ + (\alpha\delta x^2 - \beta\gamma)e_- \end{aligned}$$

where $\gamma, \delta \in \mathbb{R}$ so that

$$\text{up}(x)^2 = x^2 + 2\gamma\delta x^2 e_0 \cdot e_\infty = x^2(1 + 4\alpha\beta\gamma\delta) = 0$$

vanishes when $\alpha\beta\gamma\delta = -1/4$.

In particular, this implies $\gamma e_0 \cdot e_\infty = 2\alpha\beta\gamma = -\frac{1}{2\delta}$.

Detail. We fix the coefficient of x as unity in the expression for $\text{up}(x)$ because it is desirable to have the property that $\text{up}(x) \cdot e_i = x \cdot e_i$ for all i. That is, we would like $(\text{up}(x) \rfloor \mathbb{I}_n) \rfloor \mathbb{I}_n^{-1} = \iota(x)$. This means the up map does no scaling — it just adds on some bits involving e_0 and e_∞.

Distance from inner product. We can see from

$$\begin{aligned} \text{up}(x) \cdot \text{up}(y) &= (\gamma e_0 + x + \delta x^2 e_\infty) \cdot (\gamma e_0 + y + \delta y^2 e_\infty) \\ &= \gamma^2 \cancel{e_0^2} + \gamma \cancel{e_0 \cdot y} + \gamma\delta y^2 e_0 \cdot e_\infty \\ &+ \gamma \cancel{x \cdot e_0} + x \cdot y + \delta y^2 \cancel{x \cdot e_\infty} \\ &+ \gamma\delta x^2 e_\infty \cdot e_0 + \delta x^2 \cancel{e_\infty \cdot y} + \delta^2 x^2 y^2 \cancel{e_\infty^2} \\ &= -\frac{1}{2}(x^2 + y^2) + x \cdot y = -\frac{1}{2}(x-y)^2 \end{aligned}$$

using $e_\infty \cdot e_0 = -\frac{1}{2\gamma\delta}$ that the Euclidean distance between points is given by:

$$\|x - y\| = \sqrt{-2\,\text{up}(x) \cdot \text{up}(y)}$$

Rotors

Numerical tests for these formulas are in this notebook on CGA conventions.

Euclidean rotations. If $R \in \text{Pin}(n)$ then

$$\text{up}(Rx\tilde{R}) = \iota(R)\,\text{up}(x)\,\iota(\tilde{R})$$

where $\iota(R) \in \iota(\text{Pin}(n)) \subset \text{Pin}(n+1,1)$.

Translations. If $v \in \iota(\mathbb{R}^n)$ is a displacement vector and $\kappa = e_\infty \cdot \text{up}(0)$, then the motor

$$T(v) = \exp\left(\frac{1}{2\kappa} v e_\infty\right) = 1 + \frac{1}{2\kappa} v e_\infty$$

is a translation acting as

$$T(v)\,\text{up}(x) = \text{up}(x+v) T(v)$$

for any point $x \in \iota(\mathbb{R}^n)$.

Dilations. The dilation rotor

$$S(s) \coloneqq \exp\left(\frac{\ln s}{2\eta} e_\infty \wedge e_0\right) = \cosh\left(\frac{\ln s}{2}\right) + \sinh\left(\frac{\ln s}{2}\right)\frac{e_\infty \wedge e_0}{\eta}$$

where $\eta = e_\infty \cdot e_0$ transforms points $x \in \mathbb{R}^n$ as

$$S(s)\,\text{up}(x) = \frac{1}{s}\,\text{up}(sx) S(s)$$

where $s \in \mathbb{R}$ is a dilation factor.

Interpretation of 1-vectors

- **Planes.** Let $a = \iota(n) + \ell e_\infty$ for some $n \in \mathbb{R}^n$ and $\ell \in \mathbb{R}$. By probing,

$$\begin{aligned} a \cdot \text{up}(x) &= (\iota(n) + \ell e_\infty) \cdot (\gamma e_0 + \iota(x) + \delta x^2 e_\infty) \\ &= n \cdot x + \ell\gamma e_\infty \cdot e_0 \\ &= n \cdot x - \frac{\ell}{2\delta} \end{aligned}$$

  which vanishes when $\hat{n} \cdot x = \frac{\ell}{2\|n\|\delta}$, describing the plane with normal $\hat{n}$ a distance $\ell(2\|n\|\delta)^{-1}$ from the origin.

- **Dual spheres.** Let $a = \text{up}(b) \pm \rho^2 \delta e_\infty$. By probing,

$$\begin{aligned} a \cdot \text{up}(x) &= \text{up}(b) \cdot \text{up}(x) \pm \rho^2 \delta e_\infty \cdot \text{up}(x) \\ &= -\frac{1}{2}\|b - x\|^2 \pm \rho^2 \gamma\delta e_\infty \cdot e_0 \\ &= -\frac{1}{2}\|b - x\|^2 \mp \frac{1}{2}\rho^2 \end{aligned}$$

  which vanishes when $\|b - x\|^2 = \mp\rho^2$. When $\pm$ is in the $-$ case, this a describes a dual sphere of radius ρ, and in the $+$ case, the sphere contains no real points — it is imaginary, having an "imaginary" radius of ρ.

  Note that

$$a^2 = 2\lambda\,\text{up}(b) \cdot e_\infty = 2\lambda\gamma e_0 \cdot e_\infty = -\frac{\lambda}{\delta}$$

  so the square-radius of a dual sphere is obtained with $\lambda = -a^2\delta$ or $\rho^2 = a^2\delta$.