Conditional Gaussian

Let $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a normally distributed vector in $\mathbb{R}^D$. Suppose the space $\mathbb{R}^D = \mathbb{R}^m \oplus \mathbb{R}^n$ is split in two and write:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & R \\ R^T & \boldsymbol{\Sigma}_2 \end{bmatrix}$$

Then, $\boldsymbol{x}_1$ given $\boldsymbol{x}_2$ is distributed as:

$$\boldsymbol{x}_1 \mid \boldsymbol{x}_2 \sim \mathcal{N}(\boldsymbol{\mu}_1 + R\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_1 - R\boldsymbol{\Sigma}_2 R^T)$$

Proof. The conditional distribution $P(\boldsymbol{x}_1 \mid \boldsymbol{x}_2) = P(\boldsymbol{x}_1, \boldsymbol{x}_2)/P(\boldsymbol{x}_2)$ is also Gaussian, as it is the product of two exponentials of quadratic forms. To fully specify a Gaussian distribution, we need only find the leading coefficients of x that appear in the exponent (as mentioned in `[gaussian]`).

$$\begin{aligned} Q(\boldsymbol{x}_1) &:= -2 \ln P(\boldsymbol{x}_1 \mid \boldsymbol{x}_2) \\ &= -2 \ln P(\boldsymbol{x}_1, \boldsymbol{x}_2) + 2 \ln P(\boldsymbol{x}_2) \\ &= \begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\Sigma}_1 & R \\ R^T & \boldsymbol{\Sigma}_2 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} + \text{constant} \end{aligned}$$

The constant is independent on $\boldsymbol{x}_1$, but may depend on $\boldsymbol{x}_2$. Using results from (Rasmussen & Williams, 2008, A.3), the inverse of the block matrix is of the form

$$\begin{bmatrix} \boldsymbol{\Sigma}_1 & R \\ R^T & \boldsymbol{\Sigma}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{\boldsymbol{\Sigma}}_1 & \tilde{R} \\ \tilde{R}^T & \tilde{\boldsymbol{\Sigma}}_2 \end{bmatrix} \quad \text{where} \begin{cases} \tilde{\boldsymbol{\Sigma}}_1 &= (\boldsymbol{\Sigma}_1 - R\boldsymbol{\Sigma}_2 R^T)^{-1} \\ \tilde{R} &= -\tilde{\boldsymbol{\Sigma}}_1 R \boldsymbol{\Sigma}_2^{-1} \\ \tilde{R}^T &= -\boldsymbol{\Sigma}_2^{-1} R^T \tilde{\boldsymbol{\Sigma}}_1 \\ \tilde{\boldsymbol{\Sigma}}_2 &= \text{not important} \end{cases}$$

Using this, we may expand the quadratic form in $\boldsymbol{x}_1$ as

$$\begin{aligned} Q(\boldsymbol{x}_1) &= (\boldsymbol{x}_1 - \boldsymbol{\mu}_1)^T \tilde{\boldsymbol{\Sigma}}_1 (\boldsymbol{x}_1 - \boldsymbol{\mu}_1) + \boldsymbol{x}_1^T \tilde{R}^T (\boldsymbol{x}_2 - \boldsymbol{\mu}_2) + (\boldsymbol{x}_2 - \boldsymbol{\mu}_2)^T \tilde{R} \boldsymbol{x}_1 + \text{constant} \\ &= \boldsymbol{x}_1^T \underbrace{\tilde{\boldsymbol{\Sigma}}_1}_{\boldsymbol{\Sigma}^{-1}} \boldsymbol{x}_1 - \boldsymbol{x}^T \underbrace{\left[\tilde{\boldsymbol{\Sigma}}_1 \boldsymbol{\mu}_1 - \tilde{R}^T (\boldsymbol{x}_2 - \boldsymbol{\mu}_2)\right]}_{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} - \underbrace{\left[\boldsymbol{\mu}_1^T \tilde{\boldsymbol{\Sigma}}_1 - (\boldsymbol{x}_2 - \boldsymbol{\mu}_2)^T \tilde{R}\right]}_{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}} \boldsymbol{x}_1 \end{aligned}$$

The underbraces show the corresponding coefficients for a standard Gaussian, $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The resulting mean and covariance matrix are therefore

$$\begin{aligned} \boldsymbol{\Sigma} &= \tilde{\boldsymbol{\Sigma}}_1^{-1} \\ \boldsymbol{\mu} &= \tilde{\boldsymbol{\Sigma}}_1^{-1} \left(\tilde{\boldsymbol{\Sigma}}_1 \boldsymbol{\mu}_1 - \tilde{R}^T (\boldsymbol{x}_2 - \boldsymbol{\mu}_2)\right) \end{aligned}$$

which, using the fact that $\boldsymbol{\Sigma}_2^{-1} R^T = R\boldsymbol{\Sigma}_2^{-1}$ is symmetric, can be expressed in terms of the original block matrix components as:

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{\Sigma}_1 - R\boldsymbol{\Sigma}_2 R^T \\ \boldsymbol{\mu} &= \boldsymbol{\mu}_1 + R\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2) \end{aligned}$$

References

Rasmussen, C. E., & Williams, C. K. I. (2008). *Gaussian Processes for Machine Learning* (3. print). MIT Press.