Let $\mathcal{E}$ be a vector bundle with projection $\pi : \mathcal{E} \to \mathcal{M}$ onto base manifold $\mathcal{M}$. Let the manifold be of dimension $\dim \mathcal{M} = n$ and the fibre of dimension s, so that the total bundle is of dimension $\dim E = n + s$. Let $\mathbb{T}\,\mathcal{E}$ denote the space of tensors constructed from $\mathcal{E}$,

$$\mathbb{T}\,\mathcal{E} = \bigoplus_{r,s=0}^{\infty} \underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{r} \otimes \underbrace{\mathcal{E}^* \otimes \cdots \otimes \mathcal{E}^*}_{s}.$$

Let Γ denote the set of local sections.

Definition 1 *A* covariant derivative *is a map*

$$\mathscr{D} : \Gamma(\mathbb{T}\,\mathcal{E}) \to \Gamma(\mathbb{T}\,\mathcal{E} \otimes \mathrm{T}^*\,\mathcal{M}) \equiv \Omega^1(\mathcal{M}, \mathbb{T}\,\mathcal{E})$$

which is

1. *a derivation:*

   $$\mathscr{D}(A \otimes B) = \mathscr{D}(A) \otimes B + A \otimes \mathscr{D}(B)$$

   whenever $A \otimes B \in \Gamma(\mathbb{T}\,\mathcal{E})$; and

2. *coincident with the exterior derivative*

   $$\mathscr{D}(f) = \mathbf{d}f$$

   on scalar fields $f \in \mathbb{T}^0_0\,\mathcal{E} \cong \mathcal{C}^\infty(\mathcal{M})$.

Note that we are to view $\mathcal{C}^\infty(\mathcal{M}) \ni f$ as the one-dimensional vector bundle $\mathbb{T}^0_0\,\mathcal{E} \ni f\mathbf{1}$, identifying the scalar multiplication by f of tensors $\boldsymbol{e} \in \mathbb{T}\,\mathcal{E}$ with the *right* tensor product, $f\boldsymbol{e} \cong \boldsymbol{e} \otimes f\mathbf{1}$. This identification occurs so that we automatically have

$$\begin{aligned}\mathscr{D}(f\boldsymbol{e}) = \mathscr{D}(\boldsymbol{e} \otimes f\mathbf{1}) &= \boldsymbol{e} \otimes \mathscr{D}(f) + f\,\mathscr{D}(\boldsymbol{e}) \\ &= \boldsymbol{e} \otimes \mathbf{d}f + f\,\mathscr{D}(\boldsymbol{e}),\end{aligned}$$

which otherwise needs to be included in the definition. (Actually, no.)

Let $(\boldsymbol{e}_1, ..., \boldsymbol{e}_s)$ be a frame in the vector bundle $\mathcal{E}$ of dimension s, so that any vector field $\boldsymbol{v} \in \mathcal{E}$ has the form $\boldsymbol{v} = v^a \boldsymbol{e}_a$. Let $(\mathbf{d}x^1, ..., \mathbf{d}x^n)$ be a coframe in the cotangent space $\mathrm{T}^* \mathcal{M}$. Any element $\boldsymbol{X} \in \mathcal{E} \otimes \mathrm{T}^* \mathcal{M}$ is of the form $\boldsymbol{X} = X^a{}_\mu \boldsymbol{e}_a \otimes \mathbf{d}x^\mu$. Latin indices are for components in the vector bundle $\mathcal{E}$, and greek indices are for components in the tangent bundle $\mathbb{T}\,\mathrm{T}\,\mathcal{M} \supset \mathrm{T}^* \mathcal{M}$. In general,

$$\mathscr{D}\, \boldsymbol{e}_a = (\mathscr{D}\, \boldsymbol{e}_a)^b{}_\mu \boldsymbol{e}_b \otimes \mathbf{d}x^\mu = \boldsymbol{e}_b \otimes \boldsymbol{\theta}^b{}_a,$$

where the *connection 1-forms* are

$$\boxed{\boldsymbol{\theta}^b{}_a := (\mathscr{D}\, \boldsymbol{e}_a)^b{}_\mu \mathbf{d}x^\mu,}$$

which define the *connection coefficients* by

$$\boxed{\begin{aligned} &\mathit{\Gamma}^b{}_{\mu a} \mathbf{d}x^\mu := \boldsymbol{\theta}^b{}_a \\ \iff\ &\mathit{\Gamma}^b{}_{\mu a} = (\boldsymbol{\theta}^b{}_a)_\mu. \end{aligned}}$$

The $\mathrm{T}^* \mathcal{M}$ index (here, μ) of the connection coefficients $\mathit{\Gamma}^b{}_{\mu a}$ is known as the *differentiating index* (conventionally, the 2nd index).

Then, for a general vector $\boldsymbol{v} \in \mathcal{E}$ we have

$$\begin{aligned} \mathscr{D}\, \boldsymbol{v} = \mathscr{D}(v^a \boldsymbol{e}_a) &= \boldsymbol{e}_a \otimes (\mathscr{D}\, v^a) + v^a (\mathscr{D}\, \boldsymbol{e}_a) \\ &= \boldsymbol{e}_a \otimes \mathbf{d}v^a + v^a \boldsymbol{e}_b \otimes \boldsymbol{\theta}^b{}_a \\ &= \left(\partial_\mu v^a + \mathit{\Gamma}^a{}_{\mu b} v^b\right) \boldsymbol{e}_a \otimes \mathbf{d}x^\mu \\ &=: (\mathscr{D}_\mu\, v^a) \boldsymbol{e}_a \otimes \mathbf{d}x^\mu. \end{aligned}$$

At the cost of clarity, we may write the *covariant exterior derivative*

$$\mathscr{D} = \mathbf{d} + \underline{\boldsymbol{\theta}}$$

where $\underline{\boldsymbol{\theta}} = [\boldsymbol{\theta}^a{}_b]\boldsymbol{e}_a \otimes \boldsymbol{e}^b = [\theta^a{}_{\beta b}\mathbf{d}x^\beta]\boldsymbol{e}_a \otimes \boldsymbol{e}^b$ is a matrix (underbar) of 1-forms (bold), so that

$$\begin{aligned}
\mathscr{D}\,\vec{v} &= \mathbf{d}\vec{v} + \underline{\boldsymbol{\theta}}\vec{v} \\
(\mathscr{D}\,\vec{v})^a &= (\mathbf{d}\vec{v})^a + \boldsymbol{\theta}^a{}_b v^b \\
&= (\partial_\mu v^a + \mathit{\Gamma}^a{}_{\mu b} v^b)\,\mathbf{d}x^\mu \\
\mathscr{D}_\mu\, v^a &= \partial_\mu v^a + \mathit{\Gamma}^a{}_{\mu b} v^b.
\end{aligned}$$

A choice of covariant derivative gives a notion of parallel transport via $\mathscr{D}\,\boldsymbol{v} = 0$.

Curvature Form

$$\begin{aligned}
\underline{\boldsymbol{F}} &= \mathbf{d}\underline{\boldsymbol{\theta}} + \underline{\boldsymbol{\theta}} \wedge \underline{\boldsymbol{\theta}} \\
\boldsymbol{F}^a{}_b &= (\mathbf{d}\underline{\boldsymbol{\theta}})^a{}_b + \boldsymbol{\theta}^a{}_c \wedge \boldsymbol{\theta}^c{}_b \\
F^a{}_{b\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu &= \big(\partial_\mu \mathit{\Gamma}^a{}_{\nu b} + \mathit{\Gamma}^a{}_{\mu c}\mathit{\Gamma}^c{}_{\nu b}\big)\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \\
2F^a{}_{b[\mu\nu]} &= \partial_\mu \mathit{\Gamma}^a{}_{\nu b} - \partial_\nu \mathit{\Gamma}^a{}_{\mu b} + \mathit{\Gamma}^a{}_{\mu c}\mathit{\Gamma}^c{}_{\nu b} - \mathit{\Gamma}^a{}_{\nu c}\mathit{\Gamma}^c{}_{\mu b}
\end{aligned}$$

Notations

	general	general relativity	$U(1)$ gauge theory
fibre	$\mathbb{T}\,\mathcal{E}$	$\mathbb{T}(\mathrm{T}\,\mathcal{M})$	$\mathbb{C}$
covariant derivative	$\mathscr{D}_\mu$	∇_μ	D_μ
connection 1-form	$\boldsymbol{\theta}^a{}_b$	$\mathit{\Gamma}^a{}_{\mu b}\mathbf{d}x^\mu$	$-i\frac{e}{\hbar}A_\mu \mathbf{d}x^\mu$
curvature 2-form	$\boldsymbol{F}^a{}_b$	$\boldsymbol{R}^a{}_b = \frac{1}{2}R^a{}_{b\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$