Proof that $\det \circ \exp = \exp \circ \operatorname{tr}$

Ingredient. *Triangular matrices are closed under multiplication.* Let A and B be upper triangular matrices, so that $A_{ij} = B_{ij} = 0$ for $i > j$. From

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = \sum_k \begin{cases} A_{ik} B_{kj} & \text{if } i \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

it follows that AB is also upper triangular. In particular, $(AB)_{ii} = A_{ii} B_{ii}$.

Ingredient. *Only diagonal elements of triangular matrices affect the trace.*

$$\operatorname{tr}(AB) = \sum_k (AB)_{kk} = \sum_k A_{kk} B_{kk} = \operatorname{diag}(A) \cdot \operatorname{diag}(B)$$

Ingredient. *Only diagonal elements of triangular matrices affect the determinant.* If A is triangular, then

$$\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n A_{i\sigma(i)} = \prod_{i=1}^n A_{ii}$$

because all the permutations σ except the identity have some $1 \leq k \leq n$ such that $\sigma(k) < k$.

Ingredient. *Any square matrix A can be put in Jordan normal form $A = PJP^{-1}$, where J is upper triangular.*

$$\begin{aligned}
\det(\exp(A)) &= \det(\exp(PJP^{-1})) \\
&= \det(P \exp(J) P^{-1}) \\
&= \det(P) \det(\exp(J)) \det(P^{-1}) \\
&= \det(\exp(J)) \\
&= \prod_{i=1}^n \exp(J)_{ii} \\
&= \prod_{i=1}^n \sum_{n=0}^\infty \frac{1}{n!} (J^n)_{ii} \\
&= \prod_{i=1}^n \sum_{n=0}^\infty \frac{1}{n!} (J_{ii})^n \\
&= \prod_{i=1}^n \exp(J_{ii}) \\
&= \exp\left(\sum_{i=1}^n J_{ii} \right) \\
&= \exp(\operatorname{tr}(J)) \\
&= \exp(\operatorname{tr}(PP^{-1}J)) \\
&= \exp(\operatorname{tr}(PJP^{-1})) \\
&= \exp(\operatorname{tr}(A))
\end{aligned}$$