

The Faddeev–LeVerrier algorithm

The Faddeev–LeVerrier algorithm may be used to determine the inverse, determinant, and characteristic polynomial of an $n \times n$ matrix. The algorithm terminates in n steps, where each step involves a single matrix multiplication and only integer division. It works like magic!

FADDEEV-LEVERRIER ALGORITHM

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1 given an  $n \times n$  matrix  $A$ 
2  $c_n := 1$ 
3  $N \leftarrow 0$ 
4 for  $k \in (n - 1, \dots, 1, 0)$ 
5    $N \leftarrow N + c_{k+1}\mathbb{I}$ 
6    $c_k := \frac{1}{k-n} \text{tr}(AN)$ 
7 return
8    $A^{-1} = -N/c_0$ 
9    $\det(A) = (-1)^n c_0$ 
10   $\chi(\lambda) = \sum_{k=0}^n c_k \lambda^k$ 

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Also refer to a [Julia implementation](#).

Derivation

Start with the characteristic polynomial of A .

$$\chi(\lambda) = \det(\lambda\mathbb{I} - A) = \sum_{k=0}^n c_k \lambda^k$$

Useful fact.

The *adjunct* of a matrix, $\text{adj}(X)$, satisfies $\det(X)\mathbb{I} = A \text{adj}(X)$.

If A is $n \times n$, then $\det(X)$, and hence the entries of $X \text{adj}(X)$, are degree n polynomials in the entries of A . Hence, the entries of $\text{adj}(X)$ are degree $n - 1$ polynomials.

The entries of $N(\lambda) := \text{adj}(\lambda\mathbb{I} - A)$ are λ -polynomials of order $n - 1$, so $N(\lambda) = \sum_{k=0}^{n-1} N_k \lambda^k$ where N_k are matrices. From $\det(\lambda\mathbb{I} - A)\mathbb{I} = (\lambda\mathbb{I} - A)N(\lambda)$,

$$\begin{aligned} \det(\lambda\mathbb{I} - A)\mathbb{I} &= (\lambda\mathbb{I} - A) \sum_{k=0}^{n-1} N_k \lambda^k \\ &= -AN_0 + \sum_{k=1}^{n-1} (N_{k-1} - AN_k) \lambda^k + N_{n-1} \lambda^n \end{aligned}$$

Equating coefficients of λ with $\chi(\lambda)\mathbb{I}$, we obtain:

$$\begin{aligned} c_0\mathbb{I} &= -AN_0 \\ c_k\mathbb{I} &= N_{k-1} - AN_k \\ c_n\mathbb{I} &= N_{n-1} \end{aligned}$$

To remember these, just write $c_k\mathbb{I} = N_{k-1} - AN_k$ for all $0 \leq k \leq n$ with the understanding that N_k vanishes outside the range $0 \leq k \leq n - 1$. Equivalently,

$$N_k = N_{k+1} - c_{k+1}\mathbb{I}$$

gives a descending recurrence relation for N_k in terms of the coefficients c_k .

Finding c_k in terms of A and N_k

This stroke of genius is due to (Hou, 1998).

Useful fact: Laplace transform of derivative.

$$\begin{aligned} \mathcal{L}\{f'(t)\}(s) &= \int_0^\infty f'(t)e^{-st} dt \\ &= f(t)e^{-st} \Big|_{t=0}^\infty + s \int_0^\infty f(t)e^{-st} dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}(s) \end{aligned}$$

Consider

$$\frac{de^{At}}{dt} = Ae^{At}$$

and perform the Laplace transform to obtain

$$-\mathbb{I} + s\mathcal{L}\{e^{At}\} = A\mathcal{L}\{e^{At}\}$$

and finally take the trace:

$$s \text{tr} \mathcal{L}\{e^{At}\} - n = \text{tr}(A\mathcal{L}\{e^{At}\}) \tag{1}$$

Useful fact: the trace of a matrix exponential in terms of eigenvalues.

If λ_i are the eigenvalues of A then $\text{tr}(A) = \sum_i \lambda_i$. Also, A can be put in Jordan normal form $A = PJP^{-1}$ where J is triangular with $\text{diag}(J) = (\lambda_1, \dots, \lambda_n)$. Since it is triangular, $\text{diag}(J^k) = (\lambda_1^k, \dots, \lambda_n^k)$.

Therefore, $\text{tr}(A^k) = \text{tr}(PJ^kP^{-1}) = \text{tr}(J^kP^{-1}P) = \text{tr}(J^k) = \sum_{i=1}^n \lambda_i^k$.

Consequently, $\text{tr}(e^{At}) = \sum_{k=0}^\infty \frac{t^k}{k!} \text{tr}(A^k) = \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{i=1}^n \lambda_i^k = \sum_{i=1}^n e^{\lambda_i t}$.

We now compute the terms in Equation 1.

$$\mathcal{L}\{e^{At}\} = \int_0^\infty e^{(A-s\mathbb{I})t} dt = (A - s\mathbb{I})^{-1} e^{(A-s\mathbb{I})t} \Big|_{t=0}^\infty = (s\mathbb{I} - A)^{-1}$$

Caution.

I'm uncomfortable with these indefinite integrals. Why should $\lim_{t \rightarrow \infty} e^{(A-s\mathbb{I})t}$ converge?

Note that from $\chi(\lambda) = \det(\lambda\mathbb{I} - A) = (\lambda\mathbb{I} - A)N(\lambda)$ we have

$$(\lambda\mathbb{I} - A)^{-1} = \frac{N(\lambda)}{\chi(\lambda)} \tag{2}$$

Let $(\lambda_1, \dots, \lambda_n)$ be the eigenvalues of A . Then $A - s\mathbb{I}$ has eigenvalues $\lambda_i - s$.

$$\text{tr} \mathcal{L}\{e^{At}\} = \int_0^\infty \text{tr} e^{(A-s\mathbb{I})t} dt = \sum_{i=1}^n \int_0^\infty e^{(\lambda_i - s)t} dt = \sum_{i=1}^n \frac{1}{s - \lambda_i}$$

Recall that the roots of the characteristic polynomial of A are its eigenvalues, so $\chi(s) = \prod_{i=1}^n (s - \lambda_i)$.

$$\text{tr} \mathcal{L}\{e^{At}\} = \sum_{i=1}^n \frac{d}{ds} \ln(s - \lambda_i) = \frac{d}{ds} \ln \left(\prod_{i=1}^n (s - \lambda_i) \right) = \frac{d}{ds} \ln \chi(s) = \frac{\chi'(s)}{\chi(s)} \tag{3}$$

Substituting Equation 2 and Equation 3 into Equation 1, we have

$$\begin{aligned} s\chi'(s) - n\chi(s) &= \text{tr}(AN(\lambda)) \\ \sum_{k=0}^n (k - n)c_k s^k &= \sum_{k=0}^{n-1} \text{tr}(AN_k) s^k \end{aligned}$$

which, expanding and equating powers of λ ,

$$c_k = \frac{\text{tr}(AN_k)}{k - n}$$

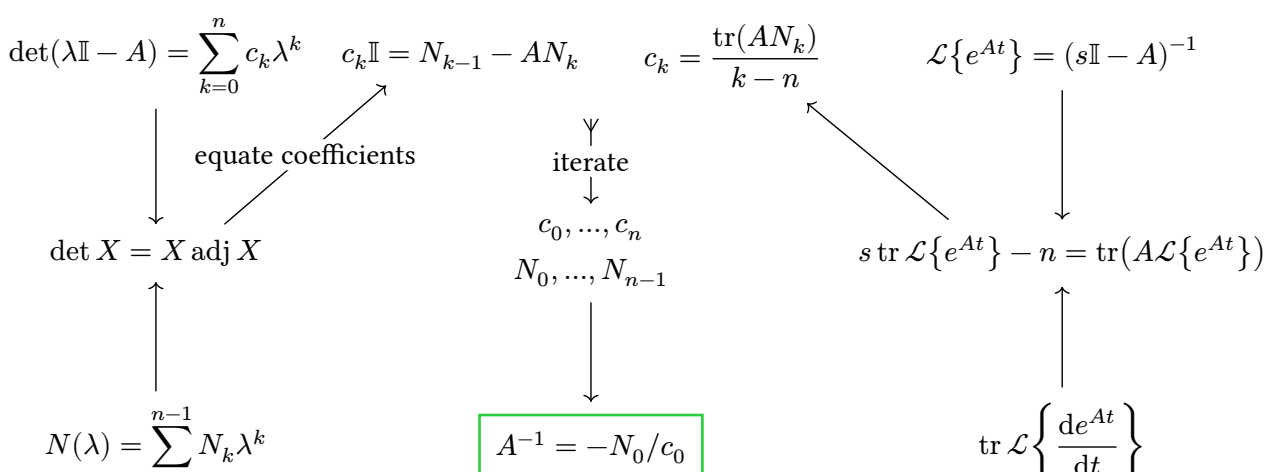
for all $0 \leq k \leq n$ where we define $N_n = 0$.

Final algorithm

Useful fact.

$$\begin{aligned} \chi(\lambda) &= c_0 + \dots + c_{n-1}\lambda^{n-1} + c_n\lambda^n \\ &= \det(-A) + \dots + \text{tr}(-A)\lambda^{n-1} + \lambda^n \\ c_0 &= (-1)^n \det(A), \quad c_{n-1} = -\text{tr}(A), \quad c_n = 1 \end{aligned}$$

Visual summary



References

Hou, S.-H. (1998). Classroom Note: A Simple Proof of the Leverrier–Faddeev Characteristic Polynomial Algorithm. *SIAM Rev.*, 40(3), 706–709. <https://doi.org/10.1137/S003614459732076X>