

Geometric algebra for &

These notes supplement an introductory talk on geometric algebra.

Introduction

Linear algebra is suited to describing *linear* transformations. Geometric algebra is suited to describing *orthogonal* transformations — rotations, reflections, rigid transformations, and so on.

The usual study of orthogonal maps from linear algebra is framed in terms of matrices. Because matrices are so general, using them to describe this subclass of linear maps implies a certain redundancy. Geometric algebra, on the other hand, is suited only to this subclass, and as a result offers a language for rigid geometry that is often very beautiful.

Definitions

Comment. Informally, geometric algebra is what you get when you define multiplication of orthonormal basis vectors $\mathbb{R}^n = \text{span}\{e_1, \dots, e_n\}$ by the rule

$$e_i e_j = -e_j e_i$$

whenever $i \neq j$ and $e_i^2 = \pm 1$, depending on the *signature* of the space.

Example. The geometric algebra of 3D Euclidean space is

$$Cl(3) = \text{span}\{1, e_1, e_2, e_3, e_2 e_3, e_3 e_1, e_1 e_2, e_1 e_2 e_3\}$$

with $e_i^2 = 1$ and therefore $(e_i e_j)^2 = -1$ and $\mathbb{I}^2 = -1$ where $\mathbb{I} := e_1 e_2 e_3$.

A more formal definition follows.

Recall. A bilinear form \cdot is a function $V \times V \rightarrow \mathbb{K}$ is

- symmetric, $u \cdot v = v \cdot u$
- linear, $(\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w$

Recall. The free tensor algebra V^{\otimes} over V has a multiplication \otimes which is

- associative, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- scalar compatible, $(\lambda a) \otimes b = \lambda(a \otimes b) = a \otimes (\lambda b)$

as well as distributive over addition.

The tensor algebra $V^{\otimes} = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$ splits into graded subspaces; an element $a \in \underbrace{V \otimes \dots \otimes V}_k$ is of *grade* k . Elements can be mixed grade!

Recall. A (two-sided) ideal in an algebra is a subspace I for which $z \in I$ implies $az, za \in I$ for any a .

An ideal behaves like a generalised zero element: anything multiplied by something zero-ish is zero-ish. Sending everything in an ideal to zero gives another well-defined algebra: this is what *taking the quotient* by an ideal means.

Define.

Let V a finite-dimensional vector space with a bilinear form \cdot . The associated *Clifford algebra* is the quotient

$$Cl(V, \cdot) \cong V^{\otimes} / I$$

where I is the ideal generated by $\{u \otimes u - u \cdot u \mid u \in V\}$.

In other words, $Cl(V, \cdot)$ is what you get if you have a free, associative algebra over V with the rule that $u^2 = u \cdot u$ for all $u \in V$.

Define. The *geometric algebra* $Cl_{p,q}$ is taken to be $Cl(\mathbb{R}^{p+q}, g)$ where g is a bilinear form with signature (p, q) .

Immediate ramifications

These facts immediately follow from the definition of a geometric algebra. Let $u, v \in V$.

Result. Vectors are invertible: $u^{-1} = u(u \cdot u)^{-1}$.

Result. The symmetrised vector product is $u \cdot v = \frac{1}{2}(uv + vu)$.

Define. Call the antisymmetric part $u \wedge v = \frac{1}{2}(uv - vu)$.

Comment. The geometric product $uv = \underbrace{u \cdot v}_{\text{scalar}} + \underbrace{u \wedge v}_{\text{bivector}}$ is of *mixed grade*.

Result. (Anti)commutativity relates to perpendicularity:

$$\begin{aligned} u \parallel v &\iff uv = vu &\iff uv = u \cdot v \\ u \perp v &\iff uv = -vu &\iff uv = u \wedge v \end{aligned}$$

Higher-grade elements

Define.

- A *k-blade* is an element of the form $u_1 u_2 \dots u_k$ where $u_i \perp u_j$ for $i \neq j$.
- A sum of k -blades is a *k-vector*.
- A sum of arbitrary blades is a *multivector*.

Comment. Not all k -vectors are blades. The simplest counterexample is $e_1 e_2 + e_3 e_4$.

Result. If $\dim V = n$, then the geometric algebra has dimension 2^n and splits into n graded subspaces:

$$\underbrace{Cl(V, g)}_{2^n} \cong \underbrace{\mathbb{R}}_{\binom{n}{0}} \oplus \underbrace{V}_{\binom{n}{1}} \oplus \underbrace{V \wedge V}_{\binom{n}{2}} \oplus \dots \oplus \underbrace{V^{\wedge n}}_{\binom{n}{n}}$$

Define. Grade projection $\langle \cdot \rangle_k : Cl(V, g) \rightarrow V^{\wedge k}$.

Result. The geometric product between a a -vector A and a b -vector B has grades:

$$AB = \underbrace{\langle AB \rangle_{|a-b|}}_{A \bullet B} + \langle AB \rangle_{|a-b|+2} + \dots + \langle AB \rangle_{a+b-2} + \underbrace{\langle AB \rangle_{a+b}}_{A \wedge B}$$

Define. All you *need* is AB and $\langle \cdot \rangle_k$, but you can define a zoo of products:

$$\begin{aligned} (\text{wedge product}) \quad A \wedge B &= \langle AB \rangle_{a+b} \\ (\text{right contraction}) \quad A \lrcorner B &= \langle AB \rangle_{a-b} \\ (\text{left contraction}) \quad A \lrcorner B &= \langle AB \rangle_{b-a} \\ (\text{commutator product}) \quad A \times B &= \frac{1}{2}(AB - BA) \end{aligned}$$

See [identities involving these products](#).

Comment. A k -blade A defines a k -dimensional subspace of V :

$$[A] := \text{span}\{u \in V \mid u \wedge A = 0\}$$

The rotation formalism

Recall. An orthogonal transformation is a linear map $A : V \rightarrow V$ which preserves the bilinear form so that $A(u) \cdot A(v) = u \cdot v$ for all $u, v \in V$.

Result. If $u \in V$ is nonzero, then

$$v \mapsto uvu^{-1}$$

is a reflection which flips every direction orthogonal to u .

Theorem. (*Cartan–Dieudonné*.) Any orthogonal transformation on V is a composition of at most $\dim V$ reflections.

Result. Any orthogonal transformation is of the form

$$u \mapsto RuR^{-1}$$

where $R = u_1 u_2 \dots u_k$ for 1-vectors u_i which we may take to be units $u_i^2 = \pm 1$.

Example. Find a rotation sending $u := 3e_1 + 2e_2$ to $v := 2e_2 + 3e_3$.

Let $\lambda := \|u\| = \|v\|$. Let $R := (1 + \frac{vu}{\lambda^2})$. It follows that $Ru = vR$ and hence $v = RuR^{-1}$. Explicitly, $R = (17 - 6e_1 e_2 - 6e_2 e_3 - 9e_1 e_3)/13$.

The magic rotor formula

Define. Define the *average conjugate* of A with over the subspace $U \subseteq G$ as

$$\mathbb{E}_U\{A\} = \frac{1}{\dim U} \sum_{g \in \mathcal{U}} gAg^{-1}$$

where \mathcal{U} is an orthogonal basis for the subspace $U \subseteq G$.

Result. $\mathbb{E}_G\{A\} = \langle A \rangle_0$ if n is even else $\mathbb{E}_G\{A\} = \langle A \rangle_0 + \langle A \rangle_n$.

Result. If $A = \langle A \rangle_q$ is a q -vector, then:

$$\mathbb{E}_{\langle G \rangle_k}\{A\} = \left[2 \binom{n}{k}^{-1} \sum_{m \text{ even}} \binom{q}{m} \binom{n-q}{k-m} - 1 \right] (-1)^{kq} A$$

Proof. See [this note](#). ■

Define. If $I = \{i_1, \dots, i_k\}$ then define multi-index notation by $e_I := e_{i_1} \dots e_{i_k}$ where $i_1 < \dots < i_k$.

Theorem. If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are orthogonal bases related by a rotation, then

$$S := \lambda \sum_{I \subseteq \{1, \dots, n\}} f_I e_I^{-1}$$

is a rotor satisfying $S e_i \tilde{S} = f_i$ where $\lambda \in \mathbb{R}$ is a normalisation factor ensuring $S \tilde{S} = 1$,

Proof. There must be some rotor R such that $f_i = R e_i \tilde{R}$. Then

$$\sum_I f_I e_I^{-1} = \sum_I R e_I \tilde{R} e_I^{-1} = 2^n R \mathbb{E}_G\{\tilde{R}\} = 2^n R \langle R \rangle_0$$

and hence $S = \lambda R$ where $\lambda 2^n \langle R \rangle_0 \in \mathbb{R}$. ■

Some topological features of rotations

Define. The (special) orthogonal groups are

$$O(V, g) = \{A \in GL(V) \mid A^* A = A A^* = \mathbb{1}\}$$

$$SO(V, g) = \{A \in O(V, g) \mid \det A = 1\}$$

where A^* is the adjoint of A with respect to the bilinear form g , satisfying $g(Au, v) = g(u, A^*v)$ for all $u, v \in V$.

Define. The *pin group* is the subgroup of elements products of unit vectors.

$$\text{Pin}(V) = \{u_1 u_2 \dots u_k \mid u_i \in V, u_i^2 = \pm 1\}$$

The *spin group*

$$\text{Spin}(V, g) = \text{Pin}(V, g) \cap Cl(V, g)^+$$

is the subgroup of even-grade elements in the pin group.

Result. $\text{Spin}(V, g)$ is a double cover of $SO(V, g)$.