Blades and the dimension of their (anti)centralizers

Theorem. If $a \in G$ is a non-(pseudo)scalar blade, then

$$\dim \mathrm{Z}_a^+(G) = \dim \mathrm{Z}_a^-(G) = \frac{1}{2}\dim G$$

where $\mathrm{Z}_a^\pm(G)$ are the (anti)centralizers of a.

Proof. By induction due to (Lasenby et al., 2024). Assume this holds for all geometric algebras in n dimensions. Let G be a geometric algebra in $n+1$ dimensions, and pick a k-blade $a \in G$ with $0 < k \leq n$. Choose a 1-vector $u \in G$ orthogonal to a in the sense that $u \rfloor a = 0$, which must be possible since a is not a pseudoscalar. Note that $ua = u \wedge a = (-1)^k a \wedge u = (-1)^k au$, which we use later.

The orthogonal subalgebra split

$$G = G^{\perp u} \oplus uG^{\perp u}$$

lets us write G in terms of the closed subalgebra $G^{\perp u} \subset G$ of elements orthogonal to u. Projecting G onto the subspaces that (anti)commute with a gives us

$$\mathrm{Z}_a^\pm(G) = \mathrm{Z}_a^\pm\left(G^{\perp u}\right) \oplus \mathrm{Z}_a^\pm\left(uG^{\perp u}\right)$$

since the (anti)centralizers $\mathrm{Z}_a^\pm$ are linear operators so distribute over the direct sum.

Remember we want to show that the dimensions of $\mathrm{Z}_a^\pm(G)$ are equal.

1. For the first term in the direct sum, since $G^{\perp u}$ is itself a geometric (sub)algebra in n dimensions, we have

$$\dim \mathrm{Z}_a^+\left(G^{\perp u}\right) = \dim \mathrm{Z}_a^-\left(G^{\perp u}\right) = \frac{1}{2}\dim G^{\perp u} = \frac{1}{4}\dim G$$

   from our inductive assumption.

2. For the second term in the direct sum, observe that if we pick some $ub \in uG^{\perp u}$ then the (anti)commutation relation $a(ub) = \pm(ub)a$ is equivalent to $ab = \pm(-1)^k ba$ because $aub = (-1)^k uab$. This means

$$ub \in \mathrm{Z}_a^\pm\left(G^{\perp u}\right) \iff b \in \begin{cases} \mathrm{Z}_a^\pm(G) \text{ if } k \text{ is even} \\ \mathrm{Z}_a^\mp(G) \text{ if } k \text{ is odd} \end{cases}$$

   or in other words that the space $\mathrm{Z}_a^\pm\left(uG^{\perp u}\right)$ is equal to $u\mathrm{Z}_a^\pm\left(G^{\perp u}\right)$ for even k and $u\mathrm{Z}_a^\mp\left(G^{\perp u}\right)$ otherwise. Either way, this is useful because it means

$$\dim \mathrm{Z}_a^\pm\left(uG^{\perp u}\right) = \frac{1}{4}\dim G$$

   since $\dim u\mathrm{Z}_a^\pm\left(G^{\perp u}\right) = \dim \mathrm{Z}_a^\pm\left(G^{\perp u}\right) = \frac{1}{4}\dim G$.

Overall, we then have

$$\dim \mathrm{Z}_a^\pm(G) = \dim \mathrm{Z}_a^\pm\left(G^{\perp u}\right) + \dim \mathrm{Z}_a^\pm\left(uG^{\perp u}\right) = \frac{1}{2}\dim G$$

which proves the induction hypothesis for $n+1$.

We need only show that the hypothesis is true for $n = 2$. For any geometric algebra G over a 2-dimensional vector space and any 1-vector $a \in G$ we have simply

$$\mathrm{Z}_a^+(G) = \mathrm{span}\{1, a\}$$
$$\mathrm{Z}_a^-(G) = \mathrm{span}\{\mathbb{I}, \mathbb{I}a\}$$

where $\mathbb{I}$ is the pseudoscalar. ∎

Propostion. If a is an n-dimensional k-blade, then

$$\dim \mathrm{Z}_a^+(G) = \begin{cases} \dim G & \text{if } k = 0 \\ \frac{1}{2}\dim G & \text{if } 0 < k < n \\ \frac{1}{2}\dim G & \text{if } k = n \text{ and } n \text{ even} \\ \dim G & \text{if } k = n \text{ and } n \text{ odd} \end{cases}$$

$$\dim \mathrm{Z}_a^-(G) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{1}{2}\dim G & \text{if } 0 < k < n \\ \frac{1}{2}\dim G & \text{if } k = n \text{ is even} \\ 0 & \text{if } k = n \text{ is odd} \end{cases}$$

Proof. Everything commutes with scalars, so the $k = 0$ case is trivial. For $0 < k < n$ we use Theorem 1. For the case $k = n$, note that the pseudoscalar $\mathbb{I} = e_1 \wedge \cdots \wedge e_n$ commutes with e_i if n is odd, and hence everything commutes with $\mathbb{I}$, and when n is even, e_i anticommutes, so that even elements commute with $\mathbb{I}$ and odd elements anticommute. ∎

References

Lasenby, A., Lasenby, J., & Matsantonis, C. (2024). Reconstructing a Rotor from Initial and Final Frames Using Characteristic Multivectors: With Applications in Orthogonal Transformations. *Mathematical Methods in the Applied Sciences*, *47*(3), 1218–1235. https://doi.org/10.1002/mma.8811