Orthogonal splits of geometric algebras

Let G be a geometric algebra. For any 1-vector u, we can define the subspaces

$$G^{\|u} := \left\{ A^{\|u} \mid A \in G \right\} = \{ A \in G \mid u \wedge A = 0 \}$$
$$G^{\perp u} := \left\{ A^{\perp u} \mid A \in G \right\} = \{ A \in G \mid u \rfloor A = 0 \}$$

where we use the multivector projections and rejections $A = A^{\|u} + A^{\perp u}$.

> **Lemma.** $G = G^{\perp u} \oplus G^{\|u}$

Proof. Any $A \in G$ is a sum of its projections and rejections $A^{\|u} \in G^{\|u}$ and $A^{\perp u} \in G^{\perp u}$.

> **Lemma.** $G^{\perp u}$ is a subalgebra.

Proof. Let $A, B \in G^{\perp u}$ which means $u \rfloor A = u \rfloor B = 0$. Since $u \rfloor$ is linear, $G^{\perp u}$ is a vector space. To show that $G^{\perp u}$ is closed under the geometric product, observe that $u \rfloor (AB) = (u \rfloor A)B + A^\star(u \rfloor B) = 0$ from the anti-derivation identity. ∎

> **Lemma.** Any element $A \in G^{\|u}$ is of the form $A = uA_\perp = u \wedge A_\perp$ where $A_\perp \in G^{\perp u}$.

Proof. For any $A \in G^{\|u}$ take $A_\perp := u^{-1}A$. To show $A_\perp \in G^{\perp u}$, write

$$u \rfloor A_\perp = u \rfloor (u^{-1}A) = (u \rfloor u^{-1})A - u^{-1}(u \rfloor A) = A - u^{-1}(uA) = 0$$

using the anti-derivation identity. Hence, $A = uA_\perp$. ∎

> **Corollary.** $G^{\|u} = uG^{\perp u} = \left\{ uA \mid A \in G^{\perp u} \right\}$

Proof. By the lemma above, the map $f(uA) = A$ is well defined for $f : G^{\|u} \to G^{\perp u}$. Since it has inverse $f^{-1}(A) = uA$ it is bijective so $f^{-1}(G^{\perp u}) = G^{\|u}$ and hence $uG^{\perp u} = G^{\|u}$.

> **Lemma.** $G = G^{\perp u} \oplus G^{\|u}$ forms a $\mathbb{Z}_2$-grading: elements multiply under the geometric product according to the multiplication table:
>
> | | $G^{\perp u}$ | $G^{\|u}$ |
> |---|---|---|
> | $G^{\perp u}$ | $G^{\perp u}$ | $G^{\|u}$ |
> | $G^{\|u}$ | $G^{\|u}$ | $G^{\perp u}$ |

Proof. We have already shown that $G^{\perp u}$ is a subalgebra, so $G^{\perp u} \times G^{\perp u} \to G^{\perp u}$ under the geometric product.

To show that $G^{\perp u} \times G^{\|u} \to G^{\|u}$, take $A \in G^{\perp u}$ and $B \in G^{\|u}$. By the previous lemma, $B = uB_\perp$ where $B_\perp \in G^{\perp u}$. We show $u \wedge (AB) = 0$ and hence $AB \in G^{\|u}$ by writing

$$u \rfloor (AB) = (u \rfloor A)uB_\perp + A^\star(u \rfloor (uB_\perp)) = A^\star u^2 B_\perp - A^\star u(u \rfloor B_\perp) = uAuB_\perp = uAB$$

where we used $u \rfloor A = 0 \iff uA = u \wedge A = A^\star \wedge u = A^\star u$. Same for $G^{\|u} \times G^{\perp u} \to G^{\|u}$.

To complete the table, pick $A, B \in G^{\|u}$ and let $A = uA_\perp$ and $B = uB_\perp$ where $A_\perp, B_\perp \in G^{\perp u}$ as before. We can show $u \rfloor (AB) = 0$ since

$$u \rfloor (uA_\perp uB_\perp) = (u \rfloor u)A_\perp uB_\perp - u(u \rfloor A_\perp)uB_\perp - uA^\star_\perp(u \rfloor u)B_\perp + uA^\star_\perp u(u \rfloor B_\perp)$$
$$= A_\perp uB_\perp - uA^\star_\perp B_\perp = A_\perp uB_\perp - A_\perp uB_\perp = 0$$

which means $AB \in G^{\perp u}$. ∎