# The hyperbolic space of univariate Gaussians

An interesting relationship exists between the space of univariate Gaussian distributions $(\mu, \sigma) \in \mathbb{R} \times (0, \infty)$ and hyperbolic geometry. This relationship can be seen with the following steps:

1. There is a natural notion of "distance" from one distribution to another, the *Kullback–Leibler divergence* $\mathrm{KL}(p : q)$, although this is not strictly a distance metric because $\mathrm{KL}(p : q) \neq \mathrm{KL}(q : p)$ in general. The divergence between two univariate Gaussians has the explicit form:

$$\mathrm{KL}(\mathcal{N}(\mu, \sigma^2) : \mathcal{N}(\nu, \rho^2)) = \log \frac{\rho}{\sigma} + \frac{\sigma^2 + (\mu - \nu)^2}{2\rho^2} - \frac{1}{2}$$

2. The divergence from $p$ to $q$ is zero when $p = q$, and positive otherwise. Thus, the first derivatives of $\mathrm{KL}(p : q)$ with respect to the parameters of $p$ vanish at the point $p = q$, but the second derivatives are positive. These positive second derivatives from a symmetric positive-definite matrix. This defines a metric tensor, known as the *Fisher information metric*, on the space of distributions. For Gaussians, this works out to be

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix} \vec{v}$$

   where $\vec{u} = (u_\mu, u_\sigma)$ and $\vec{v} = (v_\mu, v_\sigma)$ are displacement vectors for the parameters. In the style of differential geometry, this is equivalently written as

$$g = \mathrm{d}s^2 = \frac{\mathrm{d}\mu^2 + 2\,\mathrm{d}\sigma^2}{\sigma^2} \tag{1}$$

   where $g(\vec{u}, \vec{v}) = \langle \vec{u}, \vec{v} \rangle$.

3. The space of univariate Gaussian distributions equipped with the metric (1) scaled by half, $g/2$, is isometric to hyperbolic 2-space. In particular, it is isometric to one sheet of the unit hyperboloid embedded in $\mathbb{R}^3$ with the metric $\mathrm{diag}(+1, +1, -1)$.

   The isometry is most easily expressed by factoring it into a sequence of isometries between various spaces. The table below shows how to move from $(\lambda, \theta)$ coordinates parametrising the upper sheet of the unit hyperboloid $z^2 = x^2 + y^2 + 1$ to $(\mu, \sigma)$ coordinates.

| System | Metric | Description |
|---|---|---|
| $\begin{bmatrix} \lambda \\ \theta \end{bmatrix}$ | $\mathrm{d}\lambda^2 + \sinh^2 \lambda \, \mathrm{d}\theta^2$ | Surface of hyperboloid with rapidity $\lambda$ |
| $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta \sinh\lambda \\ \sin\theta \sinh\lambda \\ \cosh\lambda \end{bmatrix}$ | $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ | Cartesian hyperbolic 3-space |
| $\begin{bmatrix} \rho \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} \sinh\lambda \\ \theta \\ \cosh\lambda \end{bmatrix}$ | $\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 - \mathrm{d}z^2$ | Cylindrical hyperbolic 3-space |
| $\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{\rho}{z+1} \\ \theta \end{bmatrix}$ | $4\frac{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2}{(1 - r^2)^2}$ | Polar coordinates on hyperbolic unit disk |
| $\zeta = re^{i\theta}$ | $\frac{4\,\mathrm{d}\zeta\,\mathrm{d}\zeta^*}{(1 - \zeta\zeta^*)^2}$ | Poincaré disk |
| $\xi = \frac{1}{i}\left(\frac{\zeta + i}{\zeta - i}\right)$ | $\frac{\mathrm{d}\xi\,\mathrm{d}\xi^*}{\Im(\xi)^2}$ | Poincaré half-plane |
| $\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \begin{bmatrix} \sqrt{2}\Re(\xi) \\ \Im(\xi) \end{bmatrix}$ | $\frac{\mathrm{d}\mu^2 + 2\,\mathrm{d}\sigma^2}{2\sigma^2}$ | Parameter space of univariate Gaussians with the associated Fisher Information metric multiplied by $\frac{1}{2}$ |

See `[hyperbolic-isometries]` for numerical verifications of the relationships above.