Visualising blades by sampling points in OPNS or IPNS

Suppose we have a blade based model of geometry in which points in the *base space* $\mathbb{R}^n$ are represented by null vectors in a *representation space* $\mathbb{R}^m$. We are interested in determining which points are contained in the outer or inner product null space of a given k-blade $A \in \wedge^k \mathbb{R}^m$. Specifically, we want to sample from the sets

$$\mathsf{OPNS}(A) = \{u \in \mathbb{R}^m \mid u^2 = u \wedge A = 0\}$$
$$\mathsf{IPNS}(A) = \{u \in \mathbb{R}^m \mid u^2 = u \cdot A = 0\}$$

For example, in 3d CGA, $\boldsymbol{x} \in \mathbb{R}^3$ is represented by the vector $\mathrm{up}(\boldsymbol{x}) = n_0 + \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^2 n_\infty$ satisfying $\mathrm{up}(\boldsymbol{x})^2 = 0$, and lines may be constructed as $L = \mathrm{up}(\boldsymbol{x}) \wedge \mathrm{up}(\boldsymbol{y}) \wedge n_\infty$. We seek an algorithm to sample points $\boldsymbol{z} = t\boldsymbol{x} + (1-t)\boldsymbol{y}$ that works similarly for all blades.

Method

Given a blade $A \in \wedge^k \mathbb{R}^m$, find a vector $u \in \mathbb{R}^m$ such that $u \cdot u = 0$ and $u \wedge A = 0$ (for the OPNS*).*

1. Find a factorisation $A = \vec{a}_1 \wedge \cdots \wedge \vec{a}_k$ where $\vec{a}_i \in \mathbb{R}^m$ using the method of Fontijne & Dorst (2010), for example.

2. Form the matrix $\mathsf{A} = [\vec{a}_1 \cdots \vec{a}_k]$. If we have $u := \mathsf{A}v$ for some $v \in \mathbb{R}^k$ then u necessarily lies in the column span of A, thus $u \wedge A = 0$.

3. The condition $u \cdot u = 0$ is equivalent to $u^T \eta u = 0$ where η is the matrix of metric components. (E.g., for 3d CGA, $\eta = \mathrm{diag}(1, 1, 1, 1, -1)$.)

   Ensuring this condition holds means $u^T \eta u = v^T \mathsf{A}^T \eta \mathsf{A} v = 0$.

4. Diagonalise the symmetric matrix $\mathsf{B} := \mathsf{A}^T \eta \mathsf{A}$ so it is of the form $\mathsf{B} = \mathsf{U}^T \mathsf{D} \mathsf{U}$ where D is diagonal and $\mathsf{U}^T\mathsf{U} = \mathtt{I}$. This standard factorisation is the same as finding the eigenvalues and eigenvectors.

5. If we set $w = \mathsf{U}v$ then $u^T \eta u = v^T \mathsf{A}^T \eta \mathsf{A} v = v^T \mathsf{U}^T \mathsf{D} \mathsf{U} v = w^T \mathsf{D} w = 0$. This means the condition $u \cdot u = 0$ is equivalent to $w^T \mathsf{D} w = 0$. Finding such values for w once we knoe D is easy.

6. Let $\mathsf{D} = \mathrm{diag}(\lambda_1, ..., \lambda_k)$ and let I_+, I_-, I_0 be the sets of indices for which the corresponding entries λ_i are positive, negative, or zero, respectively.
   1. **If** $I_+ \neq \emptyset$ **and** $I_- \neq \emptyset$, then choose any values for the components w_i, but scale the components corresponding to I_+ and I_- appropriately so that

      $$\sum_{i \in I_+} w_i^2 = \sum_{j \in I_j} w_j^2$$

      which can be achieved by normalising the two subsets of components.
   2. **If** $I_+ = \emptyset$ **or** $I_- = \emptyset$, then any value w_i can be chosen for each $i \in I_0$, while all others must be zero, $j \notin I_0 \Rightarrow w_j = 0$. In particular, if $I_0 = \emptyset$, then only the trivial solution $w = 0$ exists.

   If such a w, we can recover $u = \mathsf{A}\mathsf{U}^T w$

   By varying w, subject to the constraints above (e.g., by randomly sampling w and applying normalisations) we can explore the family of solutions.

References

Fontijne, D., & Dorst, L. (2010). Efficient Algorithms for Factorization and Join of Blades. In E. Bayro-Corrochano & G. Scheuermann (Eds.), *Geometric Algebra Computing: Geometric Algebra Computing* (pp. 457–476). Springer London. https://doi.org/10.1007/978-1-84996-108-0_21