# Multivector Conjugation

**Lemma.** Conjugation by a 1-vector $u$ is a reflection. In terms of the projections and rejections,

$$uAu^{-1} = \left(A^{\perp u}\right)^\star - \left(A^{\parallel u}\right)^\star$$

for any multivector $A$.

**Proof.** Assume $A$ is a $k$-vector and then use linearity to extend to general multivectors. Using the projection and rejection to write $A = A^{\perp u} + A^{\parallel u}$, we have

$$uA^{\perp u} = u \wedge A^{\perp u} = \left(A^{\perp u}\right)^\star \wedge u = \left(A^{\perp u}\right)^\star u$$

and similarly

$$uA^{\parallel u} = u \rfloor A^{\parallel u} = \widetilde{\widetilde{A^{\parallel u}} \lfloor u} = \mathfrak{s}_{k-1}\mathfrak{s}_k A^{\parallel u} \lfloor u = -(-1)^k A^{\parallel u} \lfloor u = -\left(A^{\parallel u}\right)^\star \lfloor u = -\left(A^{\parallel u}\right)^\star u$$

where $\mathfrak{s}_k$ is the reversion sign. Summing and left-multiplying by $u^{-1}$ gives the result. ∎

**Lemma.** Conjugation by an invertible multivector $s$ is an automorphism.

**Proof.** Let $a, b \in G$ be general multivectors. then $sabs^{-1} = \left(sas^{-1}\right)\left(sbs^{-1}\right)$. ∎

In particular, this means multivector conjugation is grade-preserving.

# Kinds of reflection

We can combine incident/orthogonal components in various ways to obtain various reflections:

$$
\begin{aligned}
u^{\perp A} - u^{\parallel A} &= A^\star u A^{-1} && \text{reflect } u \text{ \underline{within} } A \\
u^{\parallel A} - u^{\perp A} &= -A^\star u A^{-1} && \text{reflect } u \text{ \underline{across} } A \\
A^{\perp u} - A^{\parallel u} &= u A^\star u^{-1} && \text{reflect } A \text{ \underline{within} } u \\
A^{\parallel u} - A^{\perp u} &= -u A^\star u^{-1} && \text{reflect } A \text{ \underline{across} } u
\end{aligned}
$$

To show these, use the definitinos from [multivector-proj-rej] and the lemma in [reversion-sign]. For example:

$$
\begin{aligned}
u^{\perp A} - u^{\parallel A} &= (u \wedge A - u \rfloor A) A^{-1} \\
&= \left( A^\star \wedge u - \widetilde{\tilde{A} \lfloor u} \right) A^{-1} \\
&= (A^\star \wedge u - \mathfrak{s}_k \mathfrak{s}_{k-1} A \lfloor u) A^{-1} \\
&= (A^\star \wedge u + A^\star \lfloor u) A^{-1} \\
&= A^\star u A^{-1}
\end{aligned}
$$