Multivector derivative identities

Lemma. If $P : G \to G$ is a projection on a geometric algebra G, then

$$P(\partial_X)X = \partial_X P(X) = \dim P(G)$$

where $P(G) = \{P(a) \mid a \in G\}$.

Proof. Without loss of generality, choose a basis $\{e_1, ..., e_N\}$ for the entire geometric algebra G such that P is defined by $P(e_i) = e_i$ for $1 \le i \le k$ and $P(e_i) = 0$ for $k < i \le N$. (If G is over a base vector space V then $N = 2^{\dim V}$.) We therefore have

$$\partial_X P(X) = \sum_{i=1}^{N} e^i P(e_i) = \sum_{i=1}^{k} 1 + \sum_{i=k+1}^{N} 0 = k$$

and similarly for $P(\partial_X)X$ if we simiply choose the reciprocal basis $\{e^i\}$ to have the desired properties instead of $\{e_i\}$. ∎

Corollary. Let $A \in G$ be a multivector and let $\dim G = 2^n$.

$$\frac{1}{2^n}\partial_X A X = \langle A\rangle_0 + \begin{cases} \langle A\rangle_n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Proof. Let $A = A_0 + \sum_i A_i + A_n$ be a sum of blades A_i with scalar part $A_0 = \langle A\rangle_0$ and pseudoscalar part $A_n = \langle A\rangle_n$. We can decompose the multivector operator ∂_X into parts which (anti)commute with a given blade.

$$\begin{aligned}
\partial_X A X &= \sum_i \partial_X A_i X \\
&= \sum_i \left\{ Z^+_{A_i}(\partial_X) + Z^-_{A_i}(\partial_X) \right\} A_i X \\
&= \sum_i A_i \left\{ Z^+_{A_i}(\partial_X) - Z^-_{A_i}(\partial_X) \right\} X \\
&= \sum_i A_i \left\{ \dim Z^+_{A_i}(G) - \dim Z^-_{A_i}(G) \right\}
\end{aligned}$$

We use Lemma 1 to obtain $Z^\pm_{A_i}(\partial_X)X = \dim Z^\pm_{A_i}(G)$. Finally, from the note on the dimension of (anti)centralizers of blades, we know that $\dim Z^+_{A_i}(G) = \dim Z^-_{A_i}(G)$ for all blades $i \notin \{0, n\}$, and also for $i = n$ if n is even. Those terms vanish. We also have $\dim Z^+_{A_i}(G) = \dim G$ and $\dim Z^-_{A_i}(G) = 0$ for scalars $i = 0$, and also for pseudoscalars $i = n$ if n is odd. Since $\dim G = 2^n$ the result follows. ∎

Corollary. If $R \in G$ is an n-dimensional even multivector:

$$\partial_X R X = 2^n \langle R\rangle_0$$

Proof. Special case of Corollary 2. ∎

Lemma. For a k-blade A and 1-vector u,

$$\partial_u A u = (n - 2k)A^\star$$

Proof. This uses the same trick: split u into parts which (anti)commute with A, rearrange, and apply Lemma 1. The problem reduces to finding the dimensions of (anti)centralizers.

$$\begin{aligned}
\partial_u A u &= \partial_u A \left\{ Z^+_A(u) + Z^-_A(u) \right\} \\
&= \partial_u \left\{ Z^+_A(u) - Z^-_A(u) \right\} A \\
&= \left\{ \dim Z^+_A(V) - \dim Z^-_A(V) \right\} A \\
&= A \begin{cases} \dim V^{\perp A} - \dim V^{\parallel A} & \text{if } A \text{ even} \\ \dim V^{\parallel A} - \dim V^{\perp A} & \text{if } A \text{ odd} \end{cases} \\
&= A \begin{cases} (n-k) - k & \text{if } k \text{ even} \\ k - (n-k) & \text{if } k \text{ odd} \end{cases} \\
&= (n-2k)(-1)^k A = (n-2k)A^\star
\end{aligned}$$

Here, $V = \langle G\rangle_1$ is the base space of 1-vectors. If A is even, then vectors perpendicular to all directions in A commute ($\dim Z^+_A(V) = \dim V^{\perp A}$) and otherwise anticommute ($\dim Z^-_A(V) = \dim V^{\parallel A} = n - \dim V^{\perp A}$). The opposite holds for A odd. ∎

Lemma. For a k-vector A and q-vector B,

$$\partial_A B A = \left[2 \sum_{m \text{ even}} \binom{q}{m}\binom{n-q}{k-m} - \binom{n}{k} \right] (-1)^{kq} B$$

Proof.

$$\begin{aligned}
\partial_A B A &= \partial_A B \left\{ Z^+_B(A) + Z^-_B(A) \right\} \\
&= \partial_A \left\{ Z^+_B(\partial_A) - Z^-_B(\partial_A) \right\} B \\
&= \left\{ \dim Z^+_B(\langle G\rangle_k) - \dim Z^-_B(\langle G\rangle_k) \right\} B
\end{aligned}$$

The terms $\dim Z^\pm_B(\langle G\rangle_k)$ are given by the number of (anti)commiting blades.

$$\partial_A B A = \left\{ 2W_{nkq} - \binom{n}{k} \right\} (-1)^{kq} B$$

where $W_{nqk} = \sum_{m \text{ even}} \binom{q}{m}\binom{n-q}{k-m}$. ∎