# Projections and rejections of vectors by blades

Let $u$ be a 1-vector and let $A$ be a blade. We can decompose $u$ into orthogonal components $u = u^{\parallel A} + u^{\perp A}$ given by

$$u^{\parallel A} = (u \rfloor A)A^{-1} = \frac{1}{2}\left(u - A^\star u A^{-1}\right)$$
$$u^{\perp A} = (u \wedge A)A^{-1} = \frac{1}{2}\left(u + A^\star u A^{-1}\right)$$

so that $u^{\parallel A}$ is *contained in* $A$ and $u^{\perp A}$ is *orthogonal to* $A$. The expanded forms follow immediately from the vector product identities.

**Lemma.**
1) $u = u^{\parallel A} + u^{\perp A}$
2) $u^{\parallel A} \wedge A = u^{\perp A} \rfloor A = 0$
3) $\left(u^{\parallel A}\right)^{\parallel A} = u^{\parallel A}$, $\left(u^{\perp A}\right)^{\perp A} = u^{\perp A}$, $\left(u^{\perp A}\right)^{\parallel A} = \left(u^{\parallel A}\right)^{\perp A} = 0$

**Proof.** The first part is trivial. For 2), assume $A$ is a $k$-blade and write:

$$u^{\parallel A} \wedge A = \left\langle (u \rfloor A)A^{-1}A \right\rangle_{k+1} = \langle u \rfloor A \rangle_{k+1} = 0$$
$$u^{\perp A} \rfloor A = \left\langle (u \wedge A)A^{-1}A \right\rangle_{k-1} = \langle u \wedge A \rangle_{k-1} = 0$$

To show that these are projections, note that

$$\begin{aligned}(u^\pm)^\pm &= \frac{1}{2}\left(u^\pm \pm A^\star u^\pm A^{-1}\right) = \frac{1}{4}\left(u \pm A^\star u A^{-1} \pm A^\star u A^{-1} + A^\star A^\star u A^{-1} A^{-1}\right) \\ &= \frac{1}{2}\left(u \pm A^\star u A^{-1}\right) = u^\pm\end{aligned}$$

where here we write $u^+ \equiv u^{\perp A}$ and $u^- \equiv u^{\parallel A}$. Lastly,

$$(u^\pm)^\mp = \frac{1}{2}\left(u^\pm \mp A^\star u^\pm A^{-1}\right) = \frac{1}{4}\left(u \pm A^\star u A^{-1} \mp A^\star u A^{-1} - A^\star A^\star u A^{-1} A^{-1}\right) = 0$$

which shows that $u^{\parallel A}$ and $u^{\perp A}$ are orthogonal projections.

# Projections and rejections of multivectors by vectors

Let $u$ be a 1-vector. We can decompose any multivector $A$ into orthogonal components $A = A^{\parallel u} + A^{\perp u}$ given by

$$\begin{aligned} A^{\parallel u} &\coloneqq (A \lfloor u)u^{-1} &= \frac{1}{2}\left(A - uA^\star u^{-1}\right) \\ A^{\perp u} &\coloneqq (A \wedge u)u^{-1} &= \frac{1}{2}\left(A + uA^\star u^{-1}\right) \end{aligned}$$

so that $A^{\parallel u}$ *contains* $u$ and $A^{\perp u}$ is *orthogonal to* $u$. The expanded forms follow immediately from the vector product identities.

**Lemma.**
1) $A = A^{\parallel u} + A^{\perp u}$
2) $u \wedge A^{\parallel u} = u \rfloor A^{\perp u} = 0$
3) $\left(A^{\parallel u}\right)^{\parallel u} = A^{\parallel u}$, $\left(A^{\perp u}\right)^{\perp u} = A^{\perp u}$, $\left(A^{\perp u}\right)^{\parallel u} = \left(A^{\parallel u}\right)^{\perp u} = 0$

**Proof.** The first part is trivial. For 2), assume $A$ is of grade $k$ and write:

$$\begin{aligned} A^{\parallel u} \wedge u &= \left\langle (A \lfloor u)u^{-1}u \right\rangle_{k+1} &= \langle A \lfloor u \rangle_{k+1} &= 0 \\ A^{\perp u} \lfloor u &= \left\langle (A \wedge u)u^{-1}u \right\rangle_{k-1} &= \langle A \wedge u \rangle_{k-1} &= 0 \end{aligned}$$

To show that these are projections, note that

$$\begin{aligned} \left(A^{\parallel u}\right)^{\parallel u} &= u \wedge \left(u^{-1} \rfloor \left(u \wedge \left(u^{-1} \rfloor A\right)\right)\right) \\ &= u \wedge \left(u^{-1} \rfloor u\right) \wedge \left(u^{-1} \rfloor A\right) - \cancel{u \wedge u} \wedge \left(u^{-1} \rfloor \left(u^{-1} \rfloor A\right)\right) = A^{\parallel u} \end{aligned}$$

using the anti-derivation identity. Since $\left(A^{\parallel u}\right)^{\perp u} = u \rfloor \left(\cancel{u^{-1} \wedge u} \wedge \left(u^{-1} \rfloor A\right)\right) = 0$, we have also $\left(A^{\perp u}\right)^{\perp u} = \left(A + A^{\parallel u}\right)^{\perp u} = A^{\perp u}$. ∎