Number of (anti)commuting blades

How many k-blades (anti)commute with a given q-blade in n dimensions? Or expressed in terms of (anti)centralizers, what is the dimension of $\mathrm{Z}_A^\pm(\langle G\rangle_k)$ for a q-blade A?

Theorem. If A is a q-blade in an n-dimensional non-degenerate geometric algebra, then

$$\dim \mathrm{Z}_A^\pm(\langle G\rangle_k) = \begin{cases} W_{nkq} & \text{if } (-1)^{kq} = \pm 1 \\ \binom{n}{k} - W_{nkq} & \text{otherwise} \end{cases} \quad \text{where} \quad W_{nkq} = \sum_{m \text{ even}} \binom{q}{m}\binom{n-q}{k-m}$$

is the number of linearly independent k-blades that (anti)commute with A.

See `[number-of-commuting-blades-tests]` for numerical verification of this result.

Proof. Without loss of generality, choose an orthonormal basis $\{e_i\}$ such that

$$A = e_1 e_2 \cdots e_q$$

and consider the k blade

$$B = e_{i_1} e_{i_2} \cdots e_{i_k}$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n$. If we count the number of basis k-blades of the form B which (anit)commute with A, we compute the dimension of the vector space $\mathrm{Z}_A^\pm(\langle G\rangle_k)$.

Let's rearrange AB into BA step by step.

First consider commuting a single vector e_i through the q-blade A, from right to left.

If e_i does not appear in A, then it anticommutes through q basis vectors, introducing a factor of $(-1)^q$. Otherwise, if e_i appears in A, it anticommutes $q-1$ one times (it commutes with itself) and introduces a factor of $(-1)^{q-1} = -(-1)^q$. In other words,

$$Ae_i = e_1 e_2 \cdots e_q e_i = e_i e_1 e_2 \cdots e_q \begin{cases} (-1)^q & \text{if } e_i \perp A \\ (-1)^{q-1} & \text{if } e_i \parallel A \end{cases} = e_i A^\star \begin{cases} +1 \text{ if } q < i \\ -1 \text{ if } i \le q \end{cases} =: (-1)^{\delta(i \le q)} e_i A^\star$$

where $\delta(i \le q)$ is 1 if $i \le q$ and 0 otherwise.

Now commute the k-blade B through A, by repeatedly applying the above.

$$\begin{aligned} AB &= A e_{i_1} e_{i_2} \cdots e_{i_k} \\ &= (-1)^q (-1)^{\delta(i_1 \le q)} e_{i_1} A e_{i_2} \cdots e_{i_k} \\ &= (-1)^{2q} (-1)^{\delta(i_1 \le q) + \delta(i_2 \le q)} e_{i_1} e_{i_2} A e_{i_3} \cdots e_{i_k} \\ &\vdots \\ &= (-1)^{kq} (-1)^{\dim(A \cap B)} BA \end{aligned}$$

Observe that the $\delta(i_j \le q)$ terms count the number of basis vectors $\{e_{i_1}, ..., e_{i_k}\}$ which appear in $A = e_1 e_2 \cdots e_q$. This is the same as the dimension of the subspace $A \cap B$ when viewing the two blades as vector subspaces.

Our original problem can now be written as

$$\begin{aligned} \dim \mathrm{Z}_A^\pm(\langle G\rangle_k) &= \dim\{B \in \langle G\rangle_k \mid AB = \pm BA\} \\ &= \dim\{B \in \langle G\rangle_k \mid (-1)^{\dim(A\cap B)} = \pm(-1)^{kq}\} \\ &= \left(\begin{array}{c} \text{number of basis } k\text{-blades } B \\ \text{where } \dim(A \cap B) \text{ is } \begin{cases} \text{even if } (-1)^{kq} = \pm 1 \\ \text{odd otherwise} \end{cases} \end{array}\right) \end{aligned}$$

Recall that $\dim(A \cap B)$ is the number of shared indices in A and B in our orthogonal basis.

This boils down to the the following combinatorics problem: How many ways W_{nkq} are there to select k indices from $\{1, 2, ..., n\}$ such that the number of indices in common with $\{1, 2, ..., q\}$ is even? The answer is

$$W_{nkq} = \sum_{m \text{ even}} \binom{q}{m}\binom{n-q}{k-m}$$

where the contributing terms are in the range $0 \le m \le \min(k, q)$.

Pulling this all together, we have

$$\dim \mathrm{Z}_A^\pm(\langle G\rangle_k) = \begin{cases} W_{nkq} & \text{if } (-1)^{kq} = \pm 1 \\ \binom{n}{k} - W_{nkq} & \text{otherwise} \end{cases}$$

where A is a q-blade. ∎