Plane-based Projective Geometric Algebra and Intrinsic/Extrinsic Orientation

In Euclidean n-space, a hyperplane A has homogeneous form

$$A : a_1 x_1 + \cdots + a_n x_n + b = 0$$

which motivates its representation as the vector

$$A \coloneqq a_1 \boldsymbol{e}_1 + \cdots + a_n \boldsymbol{e}_n + \boldsymbol{e}_0.$$

Next, k-planes can be represented as $(n-k+1)$-fold wedge products of planes of this form.

$$(n-k)\text{-plane} : A_0 \wedge \cdots \wedge A_{k-1}$$

In particular, points are wedges of n different hyperplanes.

Non-metrical duality

Later, we will want to adopt a degenerate metric. We would still like a non-degenerate duality between k- and $(n-k)$-blades, however. This means we cannot use pseudoscalar multiplication as a duality, like in conformal geometric algebra.

Introduce a metric-invariant dual operation defined by

$$\boldsymbol{e}_I \overline{\boldsymbol{e}_I} = \mathbb{I} \quad \text{(right dual)} \tag{1}$$

for any multi-index I where $\mathbb{I}$ is a *choice* of unit pseudoscalar. There are two unit pseudoscalars per algebra; let's pick $\mathbb{I} = \boldsymbol{e}_1 \boldsymbol{e}_2 \cdots \boldsymbol{e}_n \boldsymbol{e}_0$ for now.

Note that (Dorst, 2024) calls the dual defined in eq. 1 the "Hodge dual". I think this is misleading, because the Hodge dual should be metric-depentent, according to its most recognisable definition as something like $\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \omega$ or $\star A = \tilde{A}\mathbb{I}$ in GA.

I prefer to call the duality defined by eq. 1 the **right dual**, or `rdual` in code, as opposed to the **left dual** satisfying $\underline{\boldsymbol{e}_I} \boldsymbol{e}_I = \mathbb{I}$. Eric Lengyel calls these *left and right complements.*

Note that the dual $\overline{A}$ has inverse $\underline{A}$ and while $\underline{\overline{A}} = A$ always, in general we have $\overline{\overline{A}} = \underline{\underline{A}} = \pm A$ (depending on whether the grade of A and dimension of the algebra are even or odd).

Point embedding and the translation versor

We may define the point embedding map $\text{up} : \mathbb{R}^n \to R^{n+1}$ as

$$\text{up}(\vec{x}) = \overline{\vec{x} + \boldsymbol{e}_0}.$$

If we pick the metric $\boldsymbol{e}_i^2 = 1$, $\boldsymbol{e}_0^2 = 0$, then the translation versor $\mathsf{T}_{\vec{p}}[\text{up}(\vec{x})] = \text{up}(\vec{p} + \vec{x})$ is given by

$$\mathsf{T}_{\vec{p}} = \exp\left(\tfrac{1}{2}\vec{p}\boldsymbol{e}_0\right).$$

Left/right duals do not commute with versors

Note that $\mathsf{T}_{\vec{p}}\left[\overline{A}\right] \neq \overline{\mathsf{T}_{\vec{p}}[A]}$ in general, even though $\mathsf{T}_{\vec{p}}$ is an algebra automorphism. This is because left/right duality is not expressible with "pure GA" operations.

Compare this to CGA, where the dual operation is a composition of geometric multiplication (with the pseudoscalar) and possibly other automorphisms (usually reversion).

Orientation

In (Dorst, 2024), Leo introduces a cute notation for *intrinsic* and *extrinsic* notation. Here we consider the case for basis blades of grade one and two in three dimensions only.

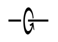
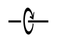
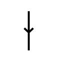
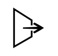
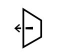
Blade	Intrinsic	Extrinsic
e_1		
$-e_1$		
e_2		
$-e_2$		
e_3		
$-e_3$		
e_{12}		
$-e_{12}$		
e_{13}		
$-e_{13}$		
e_{23}		
$-e_{23}$		

Table 1: Icons for basis blades in $\mathbb{R}^3$ with intrinsic/extrinsic orientation.

If A and B are intrinsically oriented blades, the "correct" formula for B reflected in A is

$$A[B] = (-1)^{\#A\#B} ABA^{-1}$$

derived from the vector reflection formula $b \mapsto -aba^{-1}$. For each of the $\#B$ vector factors in B, there are $\#A$ many reflections, so $\#A\#B$ many sign flips in total.

Reflection experiments

Table 2: Cayley table for $A[B] \coloneqq (-1)^{\#A\#B} ABA^{-1}$

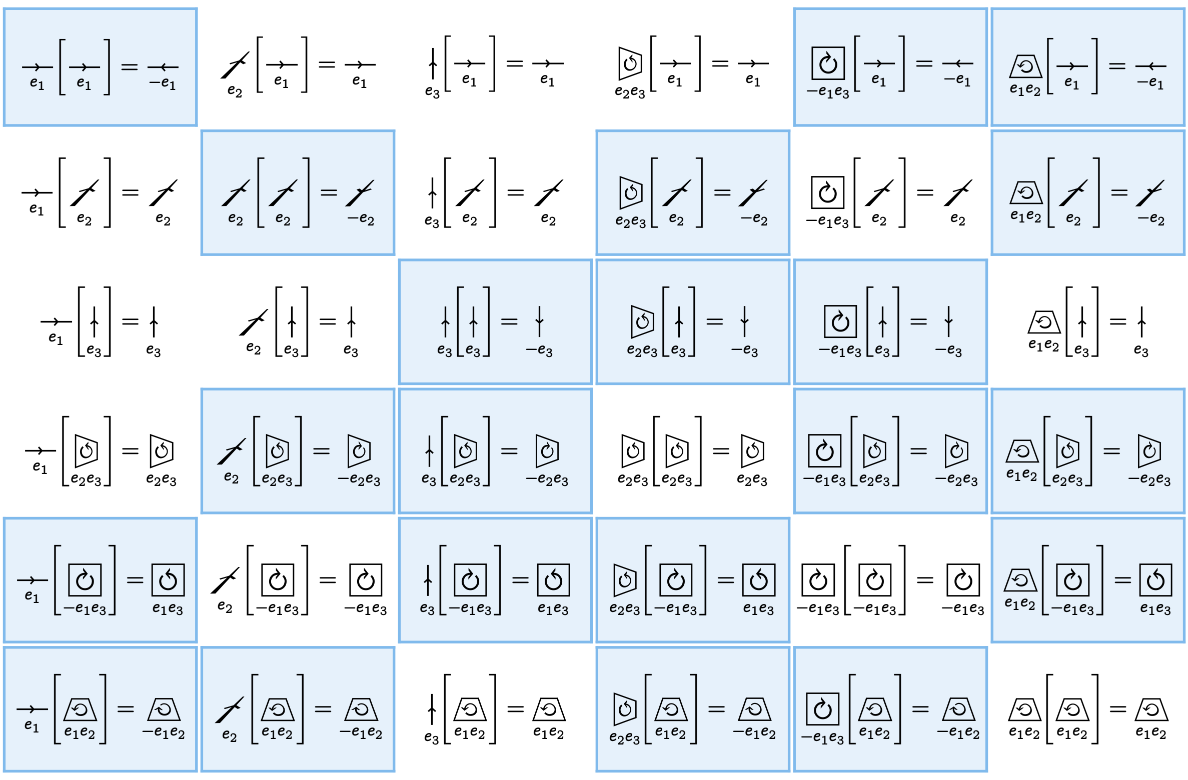


Table 3: Cayley table for $A[B] \coloneqq (-1)^{(3-\#A)\#B} ABA^{-1}$

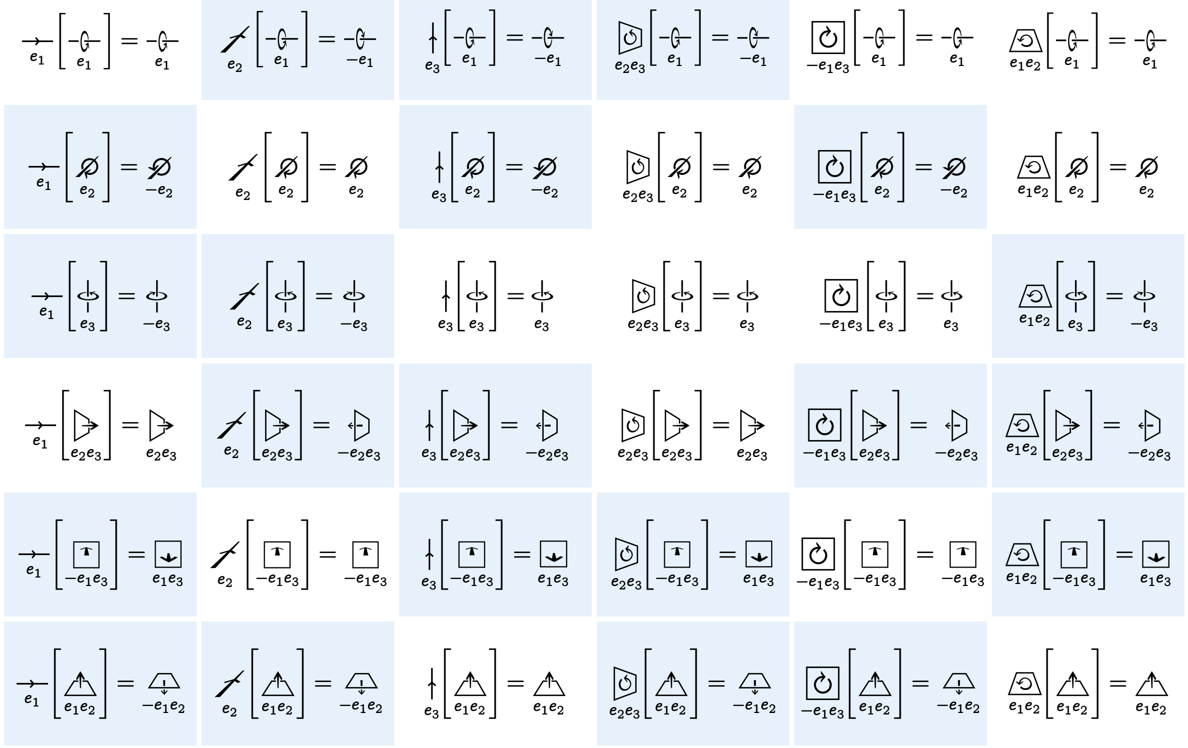


Table 4: Cayley table for $A[B] \coloneqq (-1)^{\#A(3-\#B)} ABA^{-1}$

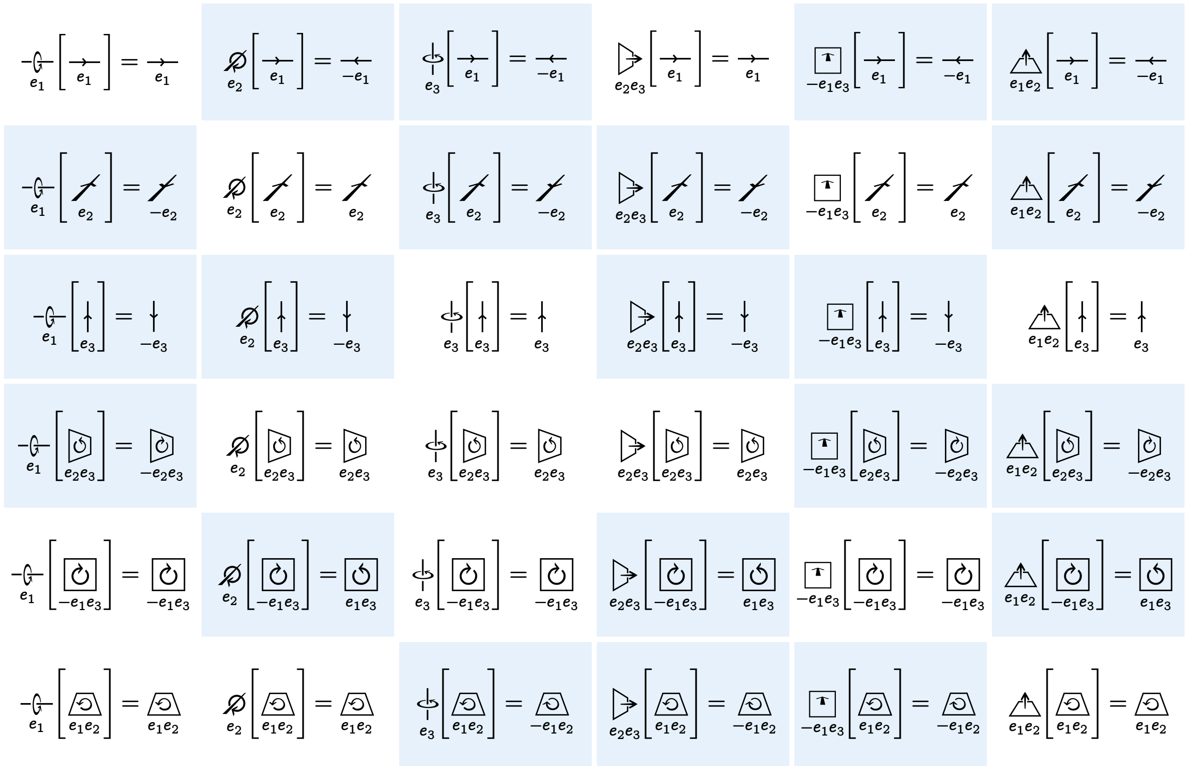
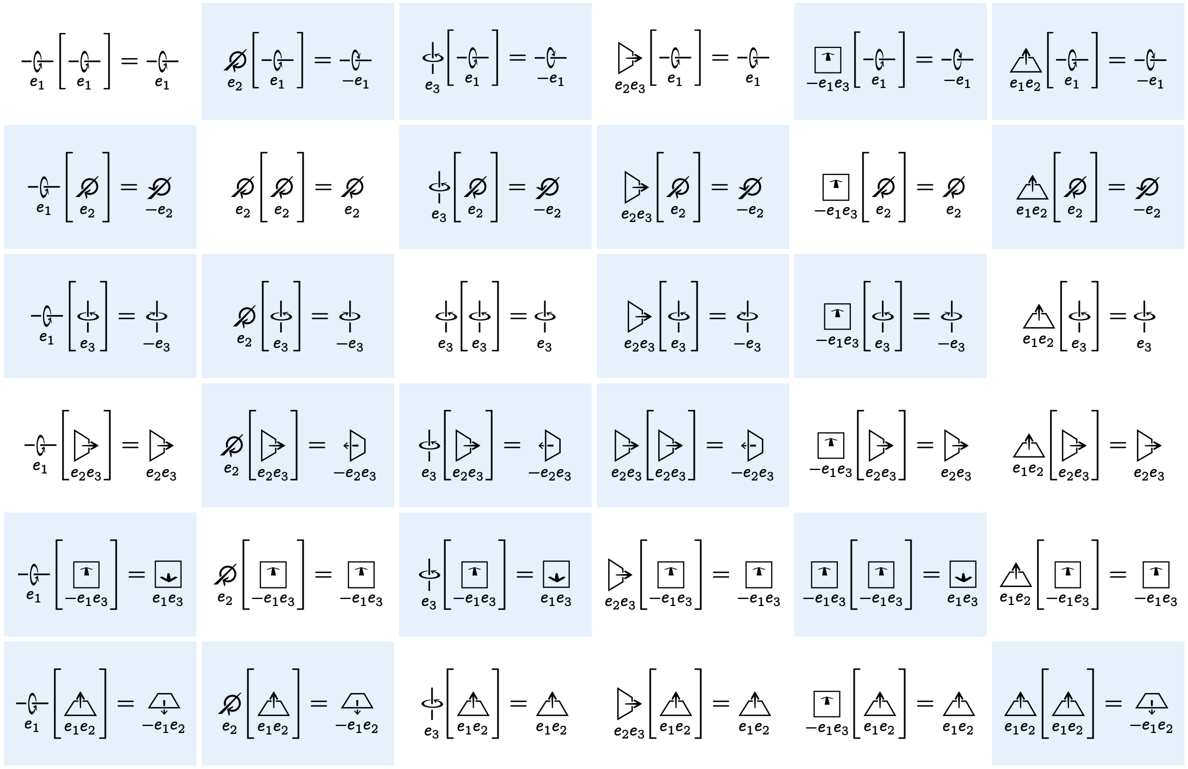


Table 5: Cayley table for $A[B] \coloneqq (-1)^{(3-\#A)(3-\#B)} ABA^{-1}$



References

Dorst, L. (2024). Projective Duality Encodes Complementary Orientations in Geometric Algebras. *Mathematical Methods in the Applied Sciences*, *47*(14), 11422–11438. https://doi.org/10.1002/mma.9754