The Poission Distribution

Discrete case

Suppose an event has a probability $p \in [0, 1]$ of occuring at each timestep k, where timesteps are equally spaced with separation Δt. Over the duration $k\Delta t \leq t < (k+n)\Delta t$, the event can occur between zero and n times. (The event cannot occur more than once in a single timestep, and different occurances are uncorrelated.)

Let $P(k, n)$ be the probability that the event occurs exactly k times in n timesteps. Let $e_1, ..., e_n \in \{0, 1\}$ be the individual outcomes at each timestep, with $e_i = 1$ if the event occured at timestep i (probability p) and $e_i = 0$ if the event did not occur (probability $1-p$). Each scenario $(e_1, ..., e_n)$ has totoal probability $p^k p^{n-k}$, where k is the number of occurances $e_i = 1$ and $k - n$ the number of non-occurances $e_i = 0$. There are $\binom{n}{k}$ such scenarios of length n with exactly k occurances. Hence,

$$P(k, n) = \binom{n}{k} p^n (1-p)^{k-n}.$$

Note that, since $P(n, k) > 0$ for any $0 \leq n \leq k$, we expect all such outcomes to have total probability of unity. By the binomial theorem,

$$\sum_{n=0}^{k} P(k, n) = \sum_{n=0}^{k} \binom{n}{k} p^n (1-p)^{k-n} = (p + (1-p))^k = 1.$$

Continuous case

In the continuous case, the event has a constant probability per unit time ρ of occuring. $P(k, t)$ is then the probability that the event occurs exactly k times within a duration $t \in \mathbb{R}$. The continuous case is obtained in the limit of the discrete case where the timestep size vanishes $\Delta t \to 0$ and the probability per unit time $\rho = p/\Delta t$ remains fixed. We have $t = k\,\Delta t$ and $p = \rho\,\Delta t$.

$$P(k, t) = \lim_{\Delta t \to 0} \binom{t/\Delta t}{n} (\rho \Delta t)^n (1 - \rho \Delta t)^{t/\Delta t - n}$$

Define $\eta = (\rho\Delta t)^{-1}$.

$$= \lim_{\eta\to\infty} \binom{t\rho\eta}{n} \left(\frac{1}{\eta}\right)^n \left(1 - \frac{1}{\eta}\right)^{t\rho\eta - n}$$

Since $\left(1 - \frac{1}{\eta}\right) \to \left(1 + \frac{1}{\eta}\right)^{-1}$ in the limit $\eta \to \infty$,

$$= \lim_{\eta\to\infty} \binom{t\rho\eta}{n} \left(\frac{1}{\eta}\right)^n \left(1 + \frac{1}{\eta}\right)^{-t\rho\eta + n}$$

$$= \lim_{\eta\to\infty} \binom{t\rho\eta}{n} \left(\frac{1}{\eta}\right)^n \left(1 + \frac{1}{\eta}\right)^n e^{-t\rho}$$

The limit of the term $\left(1 + \frac{1}{\eta}\right)^n$ is unity. Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$,

$$= \lim_{\eta\to\infty} \frac{t\rho\eta!}{n!(t\rho\eta - n)!} \left(\frac{1}{\eta}\right)^n e^{-t\rho}$$

$$= \frac{e^{-t\rho}}{n!} \lim_{\eta\to\infty} \prod_{k=0}^{n-1} (t\rho\eta - k) \left(\frac{1}{\eta}\right)^n$$

$$= \frac{e^{-t\rho}}{n!} \lim_{\eta\to\infty} \prod_{k=0}^{n-1} \frac{t\rho\eta - k}{\eta}$$

$$= \frac{e^{-t\rho}}{n!} \lim_{\eta\to\infty} \prod_{k=0}^{n-1} \left(t\rho - \frac{k}{\eta}\right)$$

$$= \frac{e^{-t\rho}}{n!} (t\rho)^n$$

$$= \frac{(t\rho)^n}{n!} e^{-t\rho}$$

$$= \frac{\lambda^n}{n!} e^{-\lambda}$$