Probabilistic ranking

Scenario

Suppose there are P players and G games, where each game is between any two (distinct) players. For the gth game, played between players A_g and B_g, the game outcome is:

$$y_g = \begin{cases} +1 \text{ if } A_g \text{ wins} \\ -1 \text{ if } B_g \text{ wins} \end{cases}$$

Model

We wish to model $\vec{y}$ by

$$y_g = \text{sign}(t_g), \quad t_g \sim \mathcal{N}\left(w_{A_g} - w_{B_g}, 1\right)$$

where to each player $p \in \{1, ..., P\}$ we assign a *skill* $w_p \in \mathbb{R}$.

Given this model, the probability of the outcomes given the players' skills is:

$$\mathrm{P}(\vec{y} \mid \vec{w}) = \prod_{g=1}^{G} \mathrm{P}(y_g \mid \vec{w})$$

$$\mathrm{P}(y_g \mid \vec{w}) = \int \mathrm{P}(y_g \mid t_g)\mathrm{P}\left(t_g \mid w_{A_g}, w_{B_g}\right) \mathrm{d}t_g$$

$$\mathrm{P}(y_g \mid t_g) = \begin{cases} 1 \text{ if } y_g = \text{sign}(t_g) \\ 0 \text{ otherwise} \end{cases}$$

$$\mathrm{P}\left(t_g \mid w_{A_g}, w_{B_g}\right) = \mathcal{N}\left(t_g \mid w_{A_g} - w_{B_g}, 1\right)$$

We can roll the last three equations into one:

$$\mathrm{P}(y_g \mid \vec{w}) = \int_0^{\infty} \mathcal{N}\left(y_g t_g \mid w_{A_g} - w_{B_g}, 1\right) \mathrm{d}t_g = \Phi\left(y_g\left(w_{A_g} - w_{B_g}\right)\right)$$

where $\Phi(x) = \int_{-\infty}^{x} \mathcal{N}(x \mid 0, 1)\, \mathrm{d}x$, leading to the final likelihood:

$$\mathrm{P}(\vec{y} \mid \vec{w}) = \prod_{g=1}^{G} \Phi\left(y_g\left(w_{A_g} - w_{B_g}\right)\right)$$

Posterior

The posterior is

$$\mathrm{P}(\vec{w} \mid \vec{y}) \propto P(\vec{y} \mid \vec{w})P(\vec{w}) = \mathcal{N}(\vec{y} \mid \mu_0, \Sigma_0) \prod_{g=1}^{G} \Phi\left(y_g\left(w_{A_g} - w_{B_g}\right)\right)$$

for a prior $P(\vec{w}) = \mathcal{N}(\vec{y} \mid \mu_0, \Sigma_0)$. This is hard to sample from.

Gibbs sampling

Introducing performance differences, $\vec{t}$

$$\mathrm{P}(\vec{w} \mid \vec{y}) = \int \mathrm{P}\left(\vec{w}, \vec{t} \mid \vec{y}\right)\mathrm{P}\left(\vec{t} \mid \vec{y}\right) \mathrm{d}\vec{t}$$

Conditioning on one player's skill, w_p

$$\mathrm{P}\left(w_p \mid \vec{y}, \vec{w}_p^{\complement}\right) \propto \mathrm{P}\left(\vec{y} \mid w_p, \vec{w}_p^{\complement}\right)\mathrm{P}\left(w_p \mid \vec{w}_p^{\complement}\right) = \mathrm{P}(\vec{y} \mid \vec{w})\mathrm{P}(w_p)$$

$$\mathrm{P}(\vec{w} \mid \vec{y}) = \int \mathrm{P}(\vec{w} \mid \vec{y}, w_p)\mathrm{P}(w_p \mid \vec{y}) \, \mathrm{d}w_p$$

Conditioning on one game's performance difference, t_g

$$\mathrm{P}(\vec{w} \mid \vec{y}) = \int \mathrm{P}(\vec{w} \mid \vec{y}, t_g)\mathrm{P}(t_g \mid \vec{y}) \, \mathrm{d}t_g$$

$$\mathrm{P}(\vec{w} \mid \vec{y}, t_g) = \mathcal{N}\left(t_g \mid y_g\left(w_{A_g} - w_{B_g}\right)\right), \quad \mathrm{P}(t_g \mid \vec{y}) = \begin{cases} 1 & \text{if } \text{sign}\, t_g = y_g \\ 0 & \text{otherwise} \end{cases}$$

Derivative of likelihood

$$\begin{aligned} \delta \log P(y \mid w) &= \sum_{j=1}^{G} \mathcal{N}\left(y_j\left(w_{A_j} - w_{B_j}\right)\right) y_j \left(\delta w_{A_j} - \delta w_{B_j}\right) \\ &= \sum_{i=1}^{P} \sum_{j=1}^{G} y_j \xi_{ij} \mathcal{N}\left(y_j\left(w_{A_j} - w_{B_j}\right)\right) \delta w_i \\ &= \delta w^T A y \end{aligned}$$

where

$$\xi_{ij} = \begin{cases} +1 \text{ if } i = A_j \\ -1 \text{ if } i = B_j \\ 0 \quad \text{otherwise} \end{cases}$$

$$A = \left[\xi_{ij} \mathcal{N}\left(y_j\left(w_{A_j} - w_{B_j}\right)\right)\right]_{ij}$$