

# Product of Gaussian probability density functions

The product of two Gaussian PDFs is an unnormalised Gaussian:

$$\mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2) \propto \mathcal{N}\left(\left(\Sigma_1 + \Sigma_2\right)^{-1}\left(\Sigma_1\mu_2 + \Sigma_2\mu_1\right), \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)^{-1}\right)$$

For many Gaussians:

$$\mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2)\cdots\mathcal{N}(\mu_N, \Sigma_N) \propto \mathcal{N}\left(\Sigma\left(\Sigma_1^{-1}\mu_1 + \cdots + \Sigma_N^{-1}\mu_N\right), \Sigma\right)$$

where  $\Sigma = \left(\Sigma_1^{-1} + \cdots + \Sigma_N^{-1}\right)^{-1}$

In other words, precisions are summed and means weighted by precisions are summed.

$$\Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \quad \text{and} \quad \Sigma^{-1}\mu = \Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2.$$

The  $D$ -dimensional Gaussian pdf with mean vector  $\mu$  and covariance matrix  $\Sigma$  is

$$\mathcal{N}(\mu, \Sigma) := \frac{1}{\sqrt{\tau^D \det(\Sigma)}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)$$

Consider the product of two  $D$ -dimensional Gaussians:

$$\mathcal{N}_{12} := \mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2) \propto \exp\left(-\frac{1}{2}(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) - \frac{1}{2}(x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2)\right)$$

The result is also a Gaussian. To find the resulting mean  $\mu$  and covariance matrix  $\Sigma$ , rearrange the exponent as

$$\begin{aligned} -2\log(\mathcal{N}_{12}) &= (x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) + (x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2) + c \\ &= x^T \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)x - x^T \left[\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2\right] - \left[\mu_1^T \Sigma_1^{-1} + \mu_2^T \Sigma_2^{-1}\right]x + c \end{aligned}$$

where  $c$  is constant with respect to  $x$ . Then compare this to the exponent of a single Gaussian:

$$\begin{aligned} -2\log(\mathcal{N}(\mu, \Sigma)) &= (x - \mu)^T \Sigma^{-1}(x - \mu) \\ &= x^T \Sigma^{-1}x - x^T \Sigma^{-1}\mu + \mu^T \Sigma^{-1}x + c \end{aligned}$$

By equating coefficients of  $x$ , we see that the resulting mean and covariance matrix are

$$\begin{aligned} \Sigma &= \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)^{-1} \\ \mu &= \Sigma \left(\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2\right) \\ &= \left(\Sigma_1 + \Sigma_2\right)^{-1}\left(\Sigma_1\mu_2 + \Sigma_2\mu_1\right) \end{aligned}$$

The last line follows from the fact that covariance matrices are symmetric, so  $\Sigma_1 \Sigma_2 \Sigma_1^{-1} = \Sigma_2$  and similarly for  $1 \leftrightarrow 2$ .