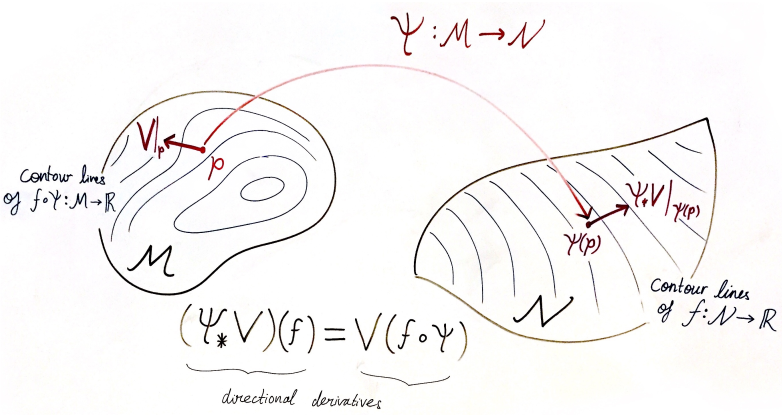
# Pushforward

If $\psi : \mathcal{M} \to \mathcal{N}$ is a smooth map between manifolds, then the *push-forward* of $\psi$, denoted $\psi_*$ or $\mathrm{T}\psi$ is the induced mapping $\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{N}$ between tangent bundles. Imagine gluing a vector $V \in \mathrm{T}_p\mathcal{M}$ to the manifold $\mathcal{M}$, and continuously moving $\mathcal{M}$ onto $\mathcal{N}$ according to $\psi$, bringing the vector with it. The final position of the vector is $\psi_* V \in \mathrm{T}_{\psi(p)}\mathcal{N}$.




**Definition 1** (Pushforward). *Let $\psi : \mathcal{M} \to \mathcal{N}$ be a smooth map. Let $V \in \mathrm{T}_p\mathcal{M}$ be a vector. The pushforward $\psi_* V \in \mathrm{T}_{\psi(p)}\mathcal{N}$ of $V$ by $\psi$ is defined by its action on a scalar field $f \in \mathcal{C}^\infty(\mathcal{N})$ via*

$$\psi_* : \mathrm{T}_p\mathcal{M} \to \mathrm{T}_{\psi(p)}\mathcal{N},$$
$$(\psi_* V)(f)\big|_{\psi(p)} = V(f \circ \psi)\big|_p.$$

*"The action of $\psi_* V$ on any function is simply the action of $V$ on the pullback of that function."* The map $\psi$ need not be invertible. The pushforward $\psi_*$ may be defined on vector fields $V \subset \mathrm{T}\mathcal{M}$ by applying the definition pointwise for all $p \in \mathcal{M}$. However, if $V$ is a vector field on $\mathcal{M}$, then $\psi_* V$ is only defined on the image $\psi(\mathcal{M}) \subseteq \mathcal{N}$ and is multi-valued where $\psi$ fails to be injective.

**Coordinate basis**

To explicitly find $\psi_*$ at a point $\psi(p) \in \mathcal{N}$, let $x^\mu$ and $y^{\bar{\mu}} = \psi^{\bar{\mu}}(x^\alpha)$ be local coordinates in a neighbourhood of $p \in \mathcal{M}$ and $\psi(p) \in \mathcal{N}$, respectively. These coordinates induce bases in the tangent spaces,

$$\mathrm{T}_p\mathcal{M} = \mathrm{span}\left\{\frac{\partial}{\partial x^\mu}\right\}_{\mu=1}^{\dim \mathcal{M}}, \quad \mathrm{T}_{\psi(p)}\mathcal{N} = \mathrm{span}\left\{\frac{\partial}{\partial y^{\bar{\mu}}}\right\}_{\bar{\mu}=1}^{\dim \mathcal{N}}.$$

Unpacking the terse definition,

$$\underbrace{\underbrace{(\psi_* \underbrace{V}_{\mathrm{T}_p\mathcal{M}})}_{\mathrm{T}_{\psi(p)}\mathcal{N}}(\underbrace{f}_{\mathcal{C}^\infty(\mathcal{N})})\Big|_{\psi(p)}}_{\mathbb{R}} = \underbrace{\underbrace{V(\underbrace{f \circ \psi}_{\mathcal{C}^\infty(\mathcal{N})})}_{\mathcal{C}^\infty(\mathcal{N})}\Big|_p}_{\mathbb{R}},$$

we proceed to write the left-hand side in coordinates,

$$(\psi_* V)(f)|_{\psi(p)} = (\psi_* V)^{\bar{\mu}} \left.\frac{\partial f}{\partial y^{\bar{\mu}}}\right|_{\psi(p)} = (\psi_*(V^\alpha \partial_\alpha))^{\bar{\mu}}|_{\psi(p)} \left.\frac{\partial f}{\partial y^{\bar{\mu}}}\right|_{\psi(p)}.$$

Equating with the right-hand side, also expressed in coordinates,

$$V(f \circ \psi)|_p = V^\nu \left.\frac{\partial f(\psi^{\bar{\alpha}}(x^\beta))}{\partial x^\nu}\right|_p = V^\nu \left.\frac{\partial f}{\partial y^{\bar{\mu}}}\right|_{\psi(p)} \left.\frac{\partial \psi^{\bar{\mu}}}{\partial x^\nu}\right|_p,$$

we cancel $\partial f / \partial y^{\bar{\mu}}|_{\psi(p)}$ from both sides and obtain

$$(\psi_*(V^\alpha \partial_\alpha))^{\bar{\mu}}|_{\psi(p)} = V^\nu \left.\frac{\partial y^{\bar{\mu}}}{\partial x^\nu}\right|_p.$$

From this we see that $\psi_*$ is linear, allowing expression as a matrix.

$$(\psi_*)^{\bar{\mu}}{}_\nu|_{\psi(p)} V^\nu = V^\nu \left.\frac{\partial y^{\bar{\mu}}}{\partial x^\nu}\right|_p$$

Finally, since $V^\mu$ is arbitrary, we may peel it off, leaving

$$(\psi_*)^{\bar{\mu}}{}_\nu|_{\psi(p)} = \left.\frac{\partial y^{\bar{\mu}}}{\partial x^\nu}\right|_p \iff \psi_* = \left.\frac{\partial y^{\bar{\mu}}}{\partial x^\nu}\right|_p \partial_{\bar{\mu}} \otimes \mathrm{d}x^\nu.$$

### Relation to Exterior Derivative

In the case that the codomain $\mathcal{N} \cong \mathbb{R}$ is one-dimensional, the pushforward of vectors in $\mathrm{T}\mathcal{M}$ by $\psi$ coincides with the exterior derivative $\mathrm{d}\psi$ of $\psi$ when viewed as a scalar field in $\mathcal{C}^\infty(\mathcal{M})$.

In coordinates, $(\psi_*)^{\bar{\mu}}{}_\nu = \partial\psi^{\bar{\mu}}/\partial x^\nu$, but $\bar{\mu}$ runs over a single coordinate since $\dim\mathbb{R} = 1$, so the index may be implicitly dropped.

$$(\psi_*)_\mu = \frac{\partial\psi}{\partial x^\mu} \iff \psi_* = \frac{\partial\psi}{\partial x^\mu}\mathrm{d}x^\mu \equiv \mathrm{d}\psi.$$

Formally, the pushforward is a linear map $\psi_* : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathbb{R}$, whereas the exterior derivative is a linear map $\mathrm{d}\psi : \mathrm{T}\mathcal{M} \to \mathbb{R}$. If $\xi$ is the single coordinate of $\mathbb{R}$, so that $\mathrm{T}\mathbb{R} = \mathrm{span}\{\partial_\xi\}$ and $\mathrm{T}^*\mathbb{R} = \mathrm{span}\{\mathrm{d}\xi\}$, then the exterior derivative may be defined in terms of the pushforward by

$$\begin{aligned}\mathrm{d}\psi(V) \coloneqq \psi_*(\mathrm{d}\xi \otimes V) &= \frac{\partial\xi}{\partial x^\nu}\left\langle\partial_\xi|\mathrm{d}\xi\right\rangle\left\langle\mathrm{d}x^\nu|V^\mu\partial_\mu\right\rangle \\ &= V^\mu\frac{\partial\xi}{\partial x^\nu}\delta^\nu_\mu = V^\mu\frac{\partial\xi}{\partial x^\mu}.\end{aligned}$$