Solution space of diagonal quadratic forms

Let $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_k)$. Consider the manifold of solutions $z \in \mathbb{R}^k$ given by:

$$z^T \Lambda z = \lambda_1 z_1^2 + \cdots + \lambda_k z_k^2 = 0$$

Let (p, q, r) be the number of entries λ_i which are positive, negative, and zero, respectively. Without loss of generality, we can focus on the special case

$$\Lambda = +\mathbb{1}_p \oplus -\mathbb{1}_q \oplus \mathbb{0}_r.$$

Given a solution for this matrix, we can obtain a solution to any diagonal matrix simply by permuting z and scaling its components $z_i \mapsto z_i/\sqrt{|\lambda_i|}$.

Define the **half n-sphere** by

$$\mathbb{H}^n := \left\{ \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \vdots \\ \sin\theta_1 \cdots \sin\theta_n \cos\theta_{n+1} \\ \sin\theta_1 \cdots \sin\theta_n \sin\theta_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \theta_i \in [0, \pi) \right\}$$

noting that $\mathbb{H}^0 = \{(1)\}$, $\mathbb{H}^1 = \{(\cos\theta, \sin\theta) \mid 0 \leq \theta < \pi\}$, etc, and $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$.

The general set of solutions is then:

$$z \in \left\{(au, \pm av, w) \mid a \in [0, \infty), u \in \mathbb{H}^{p-1}, v \in \mathbb{H}^{q-1}, w \in \mathbb{R}^r\right\}$$