What is a random variable?

σ-algebras

Let Ω be a set.

A subset $\Sigma \subseteq 2^{\Omega}$ of the powerset of Ω is a σ-**algebra** if Σ:

1. contains Ω $\qquad \Omega \in \Sigma$
2. is closed under complements $\qquad m \in \Sigma \iff m^{\complement} \in \Sigma$
3. is closed under unions $\qquad m, n \in \Sigma \implies m \cup n \in \Sigma$

It follows that Σ is closed under intersections; $m, n \in \Sigma \implies m \cap n \in \Sigma$.

Measures

Let Ω be a set and Σ be a σ-algebra over Ω.

A function $\mu : \Sigma \to [0, \infty)$ is called a **measure** if μ:

1. is non-negative $\qquad \mu(m) \geq 0$
2. additivity $\qquad \mu(m \cup n) = \mu(m) + \mu(n)$

It follows from $\mu(m + \emptyset) = \mu(m)$ that $\mu(\emptyset) = 0$.

Random variables

Let μ be a measure on the σ-algebra Σ over Ω. A **random variable** is a function

$$X : \Omega \to \Omega'$$

which induces another measure μ' on the σ-algebra Σ' over Ω'.

The new σ-algebra is defined by $\Sigma' = \{X(m) \mid m \in \Sigma\} \subseteq 2^{\Omega'}$ and the new measure by $\mu'(m) = \mu\big(X^{-1}(m)\big)$ where $X^{-1}(m) = \{n \in \Omega \mid X(n) \in m\}$ is the preimage.