Reciprocal basis

Let $\{e_i\} \subset V$ be a basis and $\cdot : V \times V \to \mathbb{R}$ a bilinear form.

A *reciprocal basis* $\{\bar{e}_i\} \subset V$ with respect to $\{e_i\}$ and $\cdot$ satisfies $e_i \cdot \bar{e}_i = \delta_{ij}$.

The basis $\{e_i\}$ need not be orthogonal.

Solving for the reciprocal basis

There exists some matrix A such that $u \cdot v = u^T A v$.

Using matrix notation, $E = (e_1 \cdots e_n)$ is an $n \times n$ matrix.

Define:

$$G = E^T A E \iff G_{ij} = e_i^T A e_j = e_i \cdot e_j$$

We want to find $\bar{E} = (\bar{e}_1 \cdots \bar{e}_n)$ so that:

$$I = E^T A \bar{E} \iff \delta_{ij} = e_i^T A \bar{e}_j = e_i \cdot \bar{e}_j$$

Solve this as $\bar{E} = (E^T A)^{-1} I = (E^T A E E^{-1})^{-1} = (G E^{-1})^{-1} = E G^{-1} \iff \bar{e}_i = e_i G^{-1}$.

$$\bar{E} = E G^{-1} \iff \bar{e}_k = e_k [e_i \cdot e_j]_{ij}^{-1}$$

See [reciprocal-basis-test] for Julia example.