Rotor of best fit formula

Definition. The rotor of best fit S which sends a frame $\{e_i\}$ to another frame $\{f_i\}$ is defined by

$$S \propto \sum_I f_I e^I$$

where the constant of proportionality is chosen so that $S\tilde{S} = 1$.

Theorem. If the target frame $f_i = Re_i\tilde{R}$ is given by a rotation of $\{e_i\}$, then the rotors R and S are equivalent in the sense that

$$f_i = Se_i\tilde{S}$$

and are equal $S = \pm R$ when $\{e_i\}$ spans the whole vector space.

To show this result,

(Lasenby et al., 2024)

With the multivector derivative, if $X \in G$ is a general multivector then $\partial_X X = \dim G$ and if P is some projection operator then $P(\partial_X)X = \dim P(G)$. In particular, if $\mathrm{Z}_A^\pm$ are the (anti)centralizer projection operators then

$$\mathrm{Z}_A^\pm(\partial_X)AX = \sum_I \mathrm{Z}_A^\pm(e^I)Ae_I = \sum_{I\in\mathrm{Z}_A^\pm} e^I A e_I = \pm A \sum_{I\in\mathrm{Z}_A^\pm} e^I e_I = \pm A \dim \mathrm{Z}_A^\pm(G)$$

where $I \in \mathrm{Z}_A^\pm$ means the multi-index ranges over basis blades $e_I \in \mathrm{Z}_A^\pm(G)$ which (anti)commute with A.

With this in mind,

$$\begin{aligned}
\partial_X RX &= \sum_k \partial_X \langle R\rangle_k X \\
&= \sum_k \left(\mathrm{Z}^+_{\langle R\rangle_k}(\partial_X) + \mathrm{Z}^-_{\langle R\rangle_k}(\partial_X)\right)\langle R\rangle_k X \\
&= \sum_k \langle R\rangle_k \left(\mathrm{Z}^+_{\langle R\rangle_k}(\partial_X) - \mathrm{Z}^-_{\langle R\rangle_k}(\partial_X)\right)X \\
&= \sum_k \langle R\rangle_k \left(\dim \mathrm{Z}^+_{\langle R\rangle_k} - \dim \mathrm{Z}^-_{\langle R\rangle_k}\right) \\
&= \dim G\langle R\rangle_0 + \begin{cases} \dim G\langle R\rangle_n & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

the last line follows from the equal dimension of (anti)centralizers for non-(pseudo)scalar blades: if a is a k-blade,

$$\dim \mathrm{Z}_a^\pm(G) = \frac{1}{2}\dim G$$

if $0 < k < n$, and for scalars

$$\dim \mathrm{Z}_1^+(G) = \dim G \quad \text{and} \quad \dim \mathrm{Z}_1^-(G) = 0$$

and finally for pseudoscalars

$$\dim \mathrm{Z}_{\mathbb{I}}^+(G) = \begin{cases} \frac{1}{2}\dim G & \text{if } n \text{ even} \\ \dim G & \text{if } n \text{ odd} \end{cases} \quad \text{and} \quad \dim \mathrm{Z}_{\mathbb{I}}^-(G) = \begin{cases} \frac{1}{2}\dim G & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Therefore, if R is even,

$$\boxed{\partial_X RX\tilde{R} = 2^n\langle R\rangle_0\tilde{R}}$$

Choosing a basis $\{e_i\}$,

$$\frac{1}{2^n}\sum_I e^I Re_I\tilde{R} = \langle R\rangle_0\tilde{R}$$

which motivates the formula

$$2^n\langle R\rangle_0\tilde{R} = \sum_I e^I f_I$$

where $f_i = Re_i\tilde{R}$

Best fit rotors

Let $\{e_1, \ldots, e_m\}$ be some initial frame and let $f_i = Re_i\tilde{R} + \varepsilon_i$ define final frame with some small noise.

The rotor of best fit S is given by

$$2^m\langle S\rangle_0\tilde{S} = \sum_I e^I f_I = 2^m\langle R\rangle_0\tilde{R} + \sum_{i=1}^m e^i\varepsilon_i + \mathcal{O}(\varepsilon^2)$$

References

Lasenby, A., Lasenby, J., & Matsantonis, C. (2024). Reconstructing a Rotor from Initial and Final Frames Using Characteristic Multivectors: With Applications in Orthogonal Transformations. *Mathematical Methods in the Applied Sciences*, *47*(3), 1218–1235. https://doi.org/10.1002/mma.8811