Blade classifications

Any Lorentzian blade is of the form

$$X = \begin{cases} \mathsf{B}_\zeta[E] & \zeta \parallel E & \text{spacelike} \\ E \wedge (\hat{n} + e) & E \perp \hat{n}, \|\hat{n}\| = 1 & \text{lightlike} \\ \mathsf{B}_\zeta[E \wedge e_0] & \zeta \perp E & \text{timelike} \end{cases}$$

for a Euclidean blade E and rapidity vector $\zeta \in \mathbb{R}^n$.

Proof.

We can always use write

$$X = X^{\perp e_0} + X^{\parallel e_0} = Y \wedge \xi, \quad \xi = \vec{x} + x_0 e_0$$

using projections and rejections where Y is a Euclidean blade and $Y \perp \vec{x}$.

1. **Spacelike case.** If $X\tilde{X} > 0$, then $\xi^2 > 0$ and by `[sta-boosts]` we have

   $$\xi = \mathsf{B}_\zeta[h] \quad h = \sqrt{\xi^2}\frac{\vec{x}}{\|\vec{x}\|}$$

   where $\tanh\zeta = x_0/\vec{x} \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$. Now, since $Y \perp \vec{x}$, we have $Y \perp \zeta$ also, and $\mathsf{B}_\zeta[Y] = Y$ so that

   $$X = Y \wedge \mathsf{B}_\zeta[h] = \mathsf{B}_\zeta[Y \wedge h] = \mathsf{B}_\zeta[E]$$

   and since $E \wedge h = 0$ we have $E \parallel \zeta$.
2. **Timelike case.** If $X\tilde{X} < 0$, then $\xi^2 < 0$ and by `[sta-boosts]` we have

   $$\xi = \mathsf{B}_\zeta\Big[\sqrt{-\xi^2}e_0\Big]$$

   where $\tanh\zeta = \vec{x}/x_0$ so that

   $$X = Y \wedge \mathsf{B}_\zeta\Big[\sqrt{-\xi^2}e_0\Big] = \mathsf{B}_\zeta[E \wedge e_0]$$

   where $E = \sqrt{-\xi^2}Y$. Since $Y \perp \vec{x}$ we have $E \perp \zeta$.

Older notes

First method.

Consider the case where X is a spacelike vector first. Then

$$X = \vec{x} + x_0 e_0$$

where $\vec{x} \in \mathbb{R}^n$ is the spatial part and x_0 is the time component, with $\|\vec{x}\|^2 > x_0^2$.

Let $s = \sqrt{X^2}$, let $\hat{n} = \vec{x}/\|\vec{x}\| \in \mathbb{R}^n$ and let $\zeta = \operatorname{arctanh}(x_0/\|\vec{x}\|)$. Then

$$\mathsf{B}_{\zeta\hat{n}}[s\hat{n}] = s\hat{n}\cosh(\zeta) + se_0\sinh(\zeta) = s\frac{\hat{n} + e_0\frac{x_0}{\|\vec{x}\|}}{\sqrt{1 - \frac{x_0^2}{\|\vec{x}\|^2}}} = s\frac{\|\vec{x}\|\hat{n} + x_0 e_0}{s} = \vec{x} + x_0 e_0 = x$$

No we may consider a spacelike blade, X. It has a decomposition

$$X = X^{\perp e_0} + X^{\parallel e_0} = Y \wedge (\vec{x} + x_0 e_0)$$

with $Y \perp \vec{x}$ which must factorise because X is a blade. You can solve for $Yx_0 = X^{\parallel e_0}e_0^{-1} = X \cdot e_0^{-1}$ and hence $\frac{\vec{x}}{x_0} = Y^{-1} \cdot X^{\perp e_0}$. Then

$$\mathsf{B}_{\zeta\hat{n}}[Y \wedge s\hat{n}] = Y \wedge \mathsf{B}_{\zeta\hat{n}}[s\hat{n}] = Y \wedge (\vec{x} + x_0 e_0) = 0$$

because $Y \perp \hat{n}$ where $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$ and $s = \sqrt{\vec{x}^2 - x_0^2}$ as before. So we have written X in terms of the Euclidean blade $E = Y \wedge s\hat{n}$ with rapidity $\zeta\hat{n}$. Note that the Euclidean blade E and rapidity vector are incident.

Second method.

If $X\tilde{X} > 0$ then X is spacelike, and we may define the timelike vector

$$\mu = \frac{1}{2}\Big(e_0 + \hat{X}e_0X^{-1}\Big) = (e_0 \wedge X)X^{-1} = e_0^{\perp X}$$

which satisfies $\mu \cdot X = 0$ and then define

$$\zeta = \operatorname{arctanh}\left(\frac{\vec{\mu}}{\mu_0}\right)$$

where $\mu = \vec{\mu} + \mu_0 e_0$ is the split into spacelike $\vec{\mu} \in \mathbb{R}^n$ and time μ_0 components. Notice that

$$\mathsf{B}_\zeta[e_0] = e_0\cosh(\zeta) + \sinh(\zeta) = \frac{e_0 + \frac{\vec{\mu}}{\mu_0}}{\sqrt{1 - \frac{\vec{\mu}^2}{\mu_0^2}}} = \frac{\mu_0 e_0 + \vec{\mu}}{\sqrt{\mu_0^2 - \vec{\mu}^2}} = \frac{\mu}{\sqrt{|\mu^2|}}$$

using identities for $\cosh \circ \operatorname{arctanh}$ and $\sinh \circ \operatorname{arctanh}$. Then,

$$E \coloneqq \mathsf{B}_{-\zeta}[X]$$

is spacelike because

$$E \cdot e_0 = \mathsf{B}_{-\zeta}\big[X \cdot \mathsf{B}_\zeta[e_0]\big] = X \cdot \frac{\mu}{\sqrt{|\mu^2|}} = 0$$

so overall we have $X = \mathsf{B}_\zeta[E]$ where E is a Euclidean blade and $\zeta \in \mathbb{R}^n$ is a rapidity vector.

It's also true that $\zeta \parallel E$, but this is harder to see this way.