Boosts in spacetime algebra

Define the boost versor as

$$\mathsf{B}_{\zeta} = \exp\left(\frac{1}{2}e_0\boldsymbol{\zeta}\right)$$

where $\boldsymbol{\zeta} \in \mathbb{R}^n$ is called the *rapidity vector* and $e_0^2 = -1$, the *relativistic velocity* $\boldsymbol{\beta} = \tanh\boldsymbol{\zeta}$ and the *gamma factor* $\gamma = \cosh\boldsymbol{\zeta} = \left(1 - \boldsymbol{\beta}^2\right)^{-\frac{1}{2}}$.

For a Euclidean vector $\boldsymbol{p} = \boldsymbol{p}^{\perp} + \boldsymbol{p}^{\parallel}$ where $\boldsymbol{p}^{\perp} \cdot \boldsymbol{\zeta} = 0$ and $\boldsymbol{p}^{\parallel} \wedge \boldsymbol{\zeta} = 0$:

$$\begin{aligned}
\mathsf{B}_{\zeta}[e_0] &= \cosh(\boldsymbol{\zeta})e_0 + \sinh(\boldsymbol{\zeta}) = \gamma(e_0 + \boldsymbol{\beta}) \\
\mathsf{B}_{\zeta}\left[\boldsymbol{p}^{\perp}\right] &= \boldsymbol{p}^{\perp} \\
\mathsf{B}_{\zeta}\left[\boldsymbol{p}^{\parallel}\right] &= \cosh(\boldsymbol{\zeta})\boldsymbol{p}^{\parallel} + \sinh(\boldsymbol{\zeta})\boldsymbol{p}^{\parallel}e_0 = \gamma\left(\boldsymbol{p}^{\parallel} + \boldsymbol{\beta}\boldsymbol{p}^{\parallel}e_0\right) \\
\mathsf{B}_{\zeta}[\boldsymbol{p}] &= \boldsymbol{p}^{\perp} + \cosh(\boldsymbol{\zeta})\boldsymbol{p}^{\parallel} + (\sinh(\boldsymbol{\zeta}) \cdot \boldsymbol{p})e_0
\end{aligned}$$

(Numerical checks)

Unboosting a vector

Consider a spacetime vector $x \in \mathbb{R}^{n,1}$

$$x = \overline{\boldsymbol{x}} + x_0 e_0$$

with a Euclidean spatial part $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ and time component $x_0 \in \mathbb{R}$. Let its norm be $\|x\| = \pm\sqrt{\left|\overline{\boldsymbol{x}}^2 - x_0^2\right|}$ to match the sign of x^2.

1. **Spacelike.** $x^2 > 0$, or $\|\overline{\boldsymbol{x}}\|/x_0 > 1$. Then

   $$\frac{x}{\|x\|} = \mathsf{B}_{\zeta}\left[\frac{\overline{\boldsymbol{x}}}{\|\overline{\boldsymbol{x}}\|}\right]$$

   where $\boldsymbol{\zeta} = \operatorname{arctanh}(x_0/\overline{\boldsymbol{x}})$.

   Proof. Note that $\boldsymbol{\beta} = x_0/\overline{\boldsymbol{x}}$ and $\gamma = \left(1 - x_0^2/\overline{\boldsymbol{x}}^2\right)^{-\frac{1}{2}} = \frac{\|\overline{\boldsymbol{x}}\|}{\|x\|}$

   $$\mathsf{B}_{\zeta}\left[\frac{\overline{\boldsymbol{x}}}{\|\overline{\boldsymbol{x}}\|}\right] = \frac{\gamma}{\|\overline{\boldsymbol{x}}\|}(\overline{\boldsymbol{x}} + \overline{\boldsymbol{x}} \cdot \boldsymbol{\beta}e_0) = \frac{1}{\|x\|}(\overline{\boldsymbol{x}} + x_0 e_0) = \frac{x}{\|x\|} \quad \blacksquare$$

2. **Timelike.** $x^2 < 0$, or $\|\overline{\boldsymbol{x}}\|/x_0 < 0$. Then

   $$\frac{x}{\|x\|} = \mathsf{B}_{\overline{\zeta}}\left[\frac{e_0}{\|e_0\|}\right]$$

   where $\overline{\boldsymbol{\zeta}} = \operatorname{arctanh}(\overline{\boldsymbol{x}}/x_0)$.

   Proof. Note that $\boldsymbol{\beta} = \overline{\boldsymbol{x}}/x_0$ and $\gamma = \left(1 - \overline{\boldsymbol{x}}^2/x_0^2\right)^{-\frac{1}{2}} = \frac{x_0}{\|x\|}$

   $$\mathsf{B}_{\zeta}\left[\frac{e_0}{\|e_0\|}\right] = \frac{\gamma}{\|e_0\|}(e_0 + \boldsymbol{\beta}) = \frac{1}{\|e_0\|}\frac{x_0 e_0 + \overline{\boldsymbol{x}}}{\|x\|} = \frac{x}{\|x\|} \quad \blacksquare$$