

# Spacetime algebra without reference to a metric signature

Define spacetime basis vectors by  $|\gamma_\mu^2| = 1$  and  $\gamma_0^2 = -\gamma_i^2$ . Define reciprocal basis vectors by  $\gamma^\mu := \gamma_\mu^{-1}$ . Define the pseudoscalar  $I := \gamma_0\gamma_1\gamma_2\gamma_3$ .

Define relative vectors  $\sigma_i := \gamma_i\gamma^0$  so that

$$\begin{aligned}\sigma_i^2 &= \gamma_i\gamma^0\gamma_i\gamma^0 \\ &= -(\gamma_i)^2(\gamma^0)^2 \\ &= -\gamma_i^2(\gamma_0^2)^{-1} \\ &= -\gamma_i^2(-\gamma_i^2)^{-1} = 1\end{aligned}$$

and

$$\begin{aligned}\sigma_1\sigma_2\sigma_3 &= \gamma_1\gamma^0\gamma_2\gamma^0\gamma_3\gamma^0 \\ &= \gamma^0\gamma_1\gamma_2\gamma_3\gamma^0\gamma^0 \\ &= \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma^0 \\ &= \gamma_0\gamma_1\gamma_2\gamma_3 = I\end{aligned}$$

Define  $\sigma^i = \sigma_i^{-1}$  and find  $\sigma^i = \gamma_0\gamma^i$  so that  $\sigma_i\sigma^i = \gamma_i\gamma^0\gamma_0\gamma^i = 1$ .

## Relation to vector cross product

If  $\times$  is the  $\mathbb{R}^3$  cross product, then for  $\vec{E} = E^i\sigma_i$  and  $\vec{B} = B^i\sigma_i$  we have

$$\begin{aligned}\langle \vec{E}\vec{B} \rangle_2 &= \sum_{i,j} E^i B^j \langle \sigma_i\sigma_j \rangle_2 \\ &= \sum_{i \neq j} E^i B^j \sigma_i\sigma_j \\ &= \sum_{i \neq j} E^i B^j \sigma^k I \varepsilon_{ijk} = (\vec{E} \times \vec{B}) I\end{aligned}$$