Variance of a sum of random variables

Let X and Y be random variables.

$$\mathrm{Var}[X] := \left\langle (X - \langle X \rangle)^2 \right\rangle = \left\langle X^2 \right\rangle - \langle X \rangle^2$$

Then:

$$\begin{aligned}
\mathrm{Var}[X+Y] &= \left\langle (X+Y)^2 \right\rangle - \langle X+Y \rangle^2 \\
&= \left\langle X^2 + 2XY + Y^2 \right\rangle - \left(\langle X \rangle^2 + 2\langle X \rangle \langle Y \rangle + \langle Y \rangle^2 \right) \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2(\langle XY \rangle - \langle X \rangle \langle Y \rangle)
\end{aligned}$$

Now, if the random variables X and Y are independent in the sense that $\rho(X, Y) = \rho(X)\rho(Y)$, then the expected value of their product is the product of their expected values.

$$\langle XY \rangle = \int XY \rho(X, Y)\, \mathrm{d}X\, \mathrm{d}Y = \int X \rho(X)\, \mathrm{d}X \int Y \rho(Y)\, \mathrm{d}Y = \langle X \rangle \langle Y \rangle$$

Hence

$$\boxed{X \perp Y \implies \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]}$$

where $X \perp Y \iff \rho(X, Y) = \rho(X)\rho(Y)$.